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PROPERTIES OF MATTER

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PREFACE

THIS book is intended primarily for the use of Physics students preparing for Higher Certificate and University Scholarship Examinations.

In addition to chapters on the various topics commonly grouped under the title Properties of Matter, a full treatment is given of those sections of post-matriculation Mechanics and Hydrostatics which are included in the Physics papers of the above examinations. A general account is given also of the Method of Virtual Work, and of the important Principles of the Conservation of Energy, of Momentum and of Angular Momentum. The inclusion of this Mechanics has seemed convenient and indeed necessary, as the treatment required for the purposes of Physics is of a more experimental character than that found in text-books on Applied Mathematics.

I have tried as far as possible to avoid duplication of matter with text-books on other branches of Physics. Thus no account is given of the Kinetic Theory of Gases, since this receives adequate treatment in books on Heat.

Many of the questions at the end of each chapter are taken from examination papers, and for permission to reproduce them I am indebted to the Northern Universities Joint Matriculation Board, the Oxford and Cambridge Schools Examination Board, the Syndics for the Cambridge Local Examinations, the Delegates for the Oxford Local Examinations, the Cambridge University Press and the Oxford University Press.

D. N. S.

LEEDS, *December*, 1936.

PREFACE TO SECOND EDITION

In this edition a chapter on Viscosity has been added.

It became evident shortly after this book made its first appearance that there was a considerable demand for a treatment of Viscosity on lines similar to the other chapters. The new chapter has been added in response to this demand, and it is hoped that the book will now fill the requirements of all students who might wish to use it.

D. N. S.

Leeds, *April*, 1937.

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ABBREVIATIONS USED FOR EXAMINATION QUESTIONS

N.U.	Northern Universities Higher Certificate.
O. and C.	Oxford and Cambridge Schools Examination Higher Certificate.
O.	Oxford Local Higher Certificate.
C.	Cambridge Local Higher Certificate.
C. Schol.	Open Scholarship, Cambridge University.
O. Schol.	Open Scholarship, Oxford University.

PROPERTIES OF MATTER

CHAPTER I

CIRCULAR MOTION

1. Motion of a Particle in a Circle with Uniform Speed

A PARTICLE is travelling with uniform speed when it covers equal lengths along its path in equal times, no matter how small these times may be. The path may be a straight line or any curve. A particle describing a straight line with uniform speed has a uniform or constant velocity. If the particle is describing a curve, then its direction of motion is continually changing and its velocity cannot be uniform.

In the case of a particle describing a circle, centre C and radius r, with uniform speed v, it is clear that the particle has an acceleration, which, from the symmetry of the case, must be constant in magnitude. In Fig. 1 let P be the particle. The number of radians through which CP turns in 1 sec. is called the angular velocity of P about C. Let ω be this angular velocity. Clearly $\omega = \frac{v}{r}$, for the particle covers a length v of the arc in 1 sec., and this subtends at C an angle $\frac{v}{r}$ radians. In the notation of the calculus $\omega = \frac{d\theta}{dt}$.

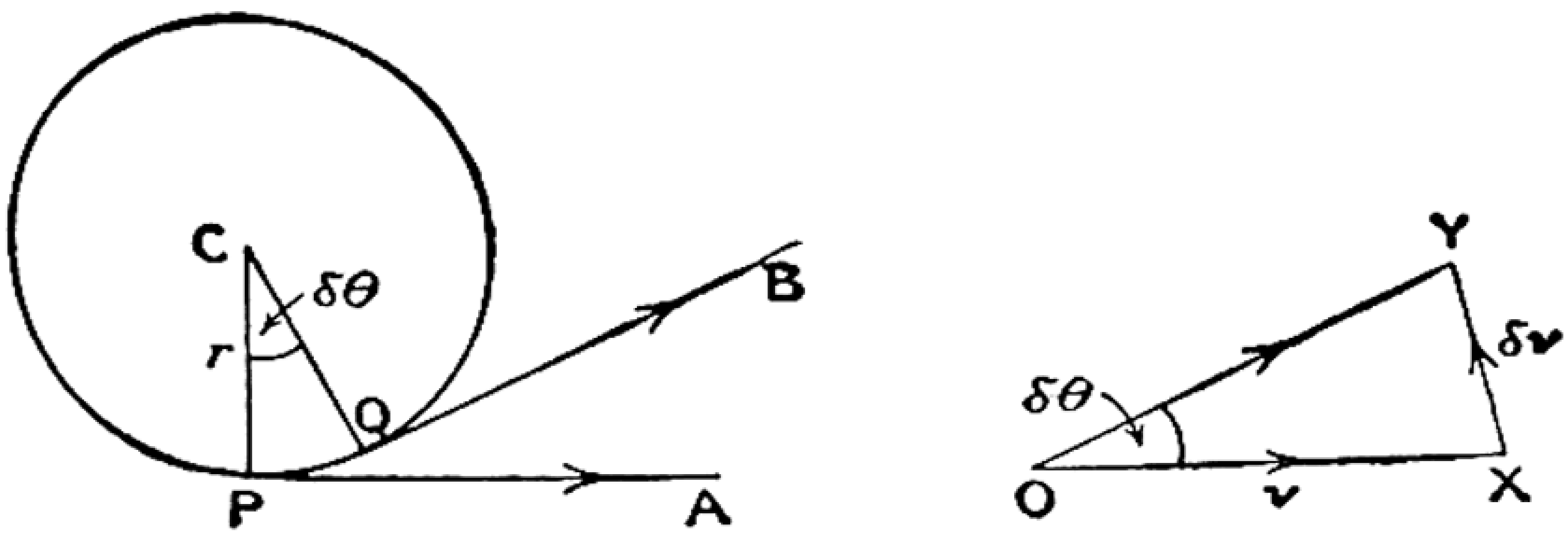

FIG. 1.

Let the particle move from P to Q in the short time δt, the small angle traced out by the radius in this time being $\delta\theta$. Let PA, QB represent the velocities of the particle when at P and Q. To find the change in velocity that has occurred, draw OX equal and parallel to PA and OY equal and parallel to QB. Then XY represents in magnitude and direction the change in velocity δv during the time δt, since XY is the velocity which must be added to or compounded with OX to obtain OY.

In magnitude $$\delta v = 2v \sin\frac{\delta\theta}{2} = v.\frac{\sin\frac{\delta\theta}{2}}{\frac{\delta\theta}{2}}.\delta\theta$$

and the acceleration $$= \frac{dv}{dt} = \mathrm{L}t\,\frac{\delta v}{\delta t} = \mathrm{L}t\,v.\frac{\sin\frac{\delta\theta}{2}}{\frac{\delta\theta}{2}}.\frac{\delta\theta}{\delta t} = v.1.\omega$$

$$= \frac{v^2}{r} = \omega^2.r.$$

Also when $\delta\theta \longrightarrow 0$ δv becomes perpendicular to OX, so the acceleration is perpendicular to the velocity and is directed towards the centre of the circle.

Force Necessary to Produce the Motion

If the particle has a mass m the force required to produce this acceleration is $\frac{m.v^2}{r}$ or $m.\omega^2.r$ and is directed towards the centre of the circle. Whenever a particle of mass m describes a circle of radius r with constant speed v the forces on it must be equivalent to a single force $\frac{m.v^2}{r}$ directed towards the centre of the circle.

Notice that the formulæ $\frac{m.v^2}{r}$ and $m.\omega^2.r$ give the force in absolute units; dynes when m is in grams, r in cms., v in cms. per sec.; poundals if m is in lb., r in ft., v in ft. per sec.

2. Motion of a Particle along a Plane Curve

Let P and Q be neighbouring points on the curve described by the particle, the velocities of the particle at P and Q being

v and v', as shown in Fig. 2. PC and QC are the normals at P and Q intersecting in C. Ultimately when P and Q are taken indefinitely close to one another C becomes the centre of curvature for the point P, and CP the radius of curvature at P. The radius of curvature is denoted by ρ.

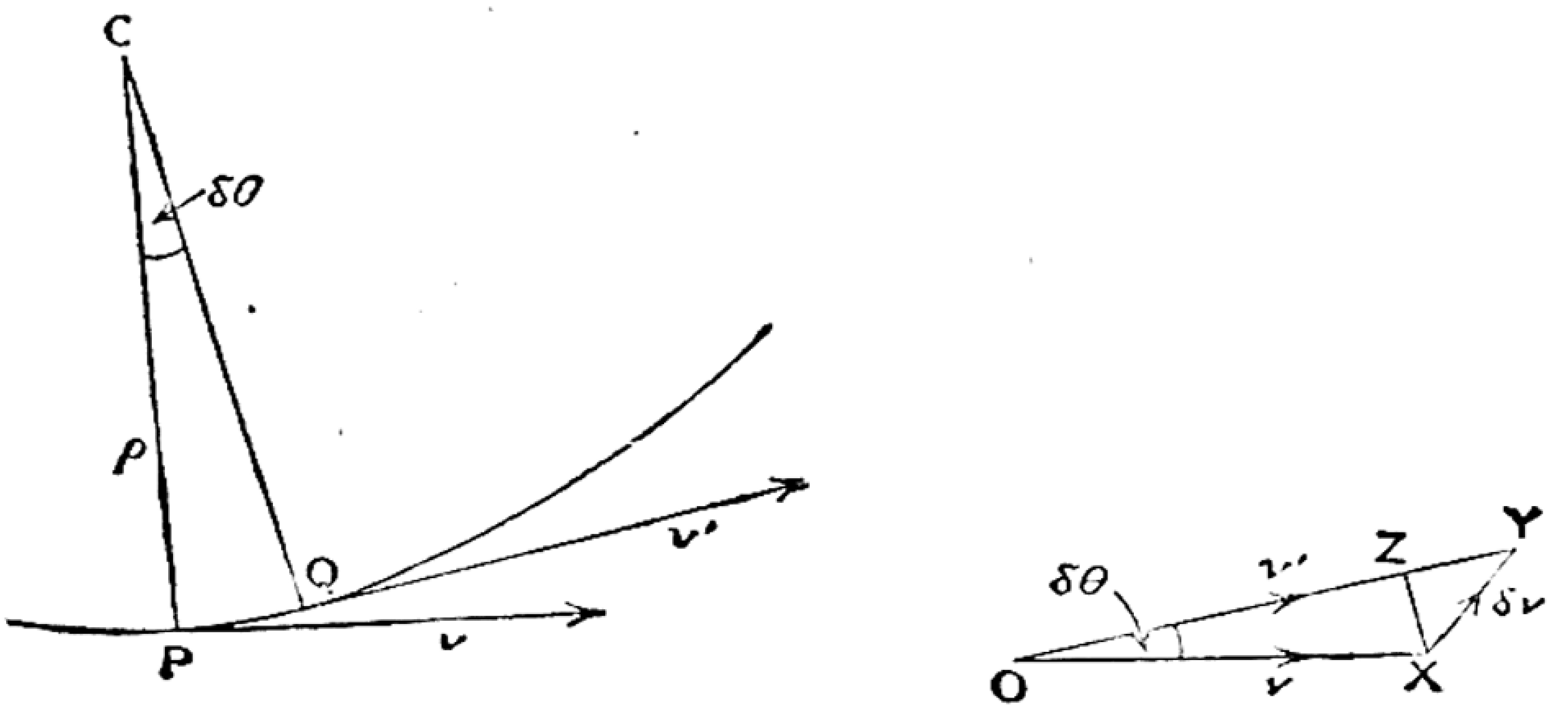

FIG. 2.

As before, OX and OY represent the velocities v and v'. The change in velocity δv now has components $XZ = v.\delta\theta$ along the normal PC and $ZY = v' - v$ along the tangent. If the time taken from P to Q is δt, then the normal acceleration is

$$\mathrm{Lt}\,\frac{v\,\delta\theta}{\delta t} = \mathrm{Lt}\,v.\frac{\delta\theta}{\delta s}.\frac{\delta s}{\delta t} = v.\frac{1}{\rho}.v = \frac{v^2}{\rho},$$

δs being the length of the small arc PQ and equal to $\rho.\delta\theta$.

The tangential acceleration is $\mathrm{Lt}\,\dfrac{v' - v}{\delta t}$. This is the rate of change of the speed of the particle in its path.

If the speed of the particle in its path is constant the tangential force at all points must be zero. The forces on the particle must be equal at all points to a single force $\dfrac{m.v^2}{\rho}$ acting at right angles to the path towards the centre of curvature, m being the mass of the particle. ρ will vary from point to point along the curve.

3. The Hodograph

Consider a particle P describing a curve with varying speed. If a vector OQ is drawn from some fixed point O to represent

the velocity of P, as P moves along the curve the point Q will trace out another curve. This locus of Q is called the hodograph of P.

In Fig. 3 the tangential lines represent the velocities of P when at P_1, P_2, P_3 . . . ; the vectors OQ_1, OQ_2, OQ_3 . . . are drawn equal and parallel to the corresponding tangential lines and also represent the velocities of P when at P_1, P_2, P_3 . . . The points Q_1, Q_2, Q_3 . . . are on the hodograph of P.

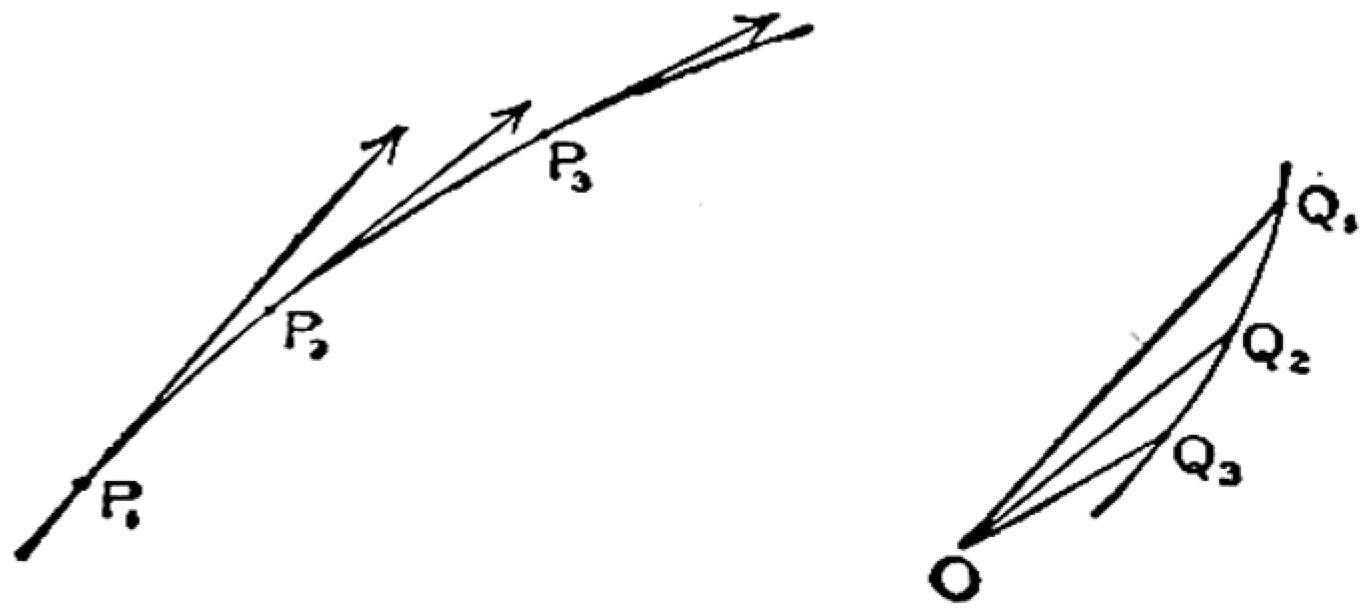

FIG. 3.

The importance of the hodograph lies in the fact that the velocity of Q in the hodograph represents in magnitude and direction the acceleration of P. To see that this is so consider the points P_1 and P_2 in Fig. 3 to be close together and the small distance P_1P_2 to be described in the small time δt. The change in velocity in time δt will be Q_1Q_2 and the acceleration of P will be

$$Lt \frac{Q_1Q_2}{\delta t} \qquad \text{when } \delta t \rightarrow O.$$

But this is the velocity of Q in the hodograph.

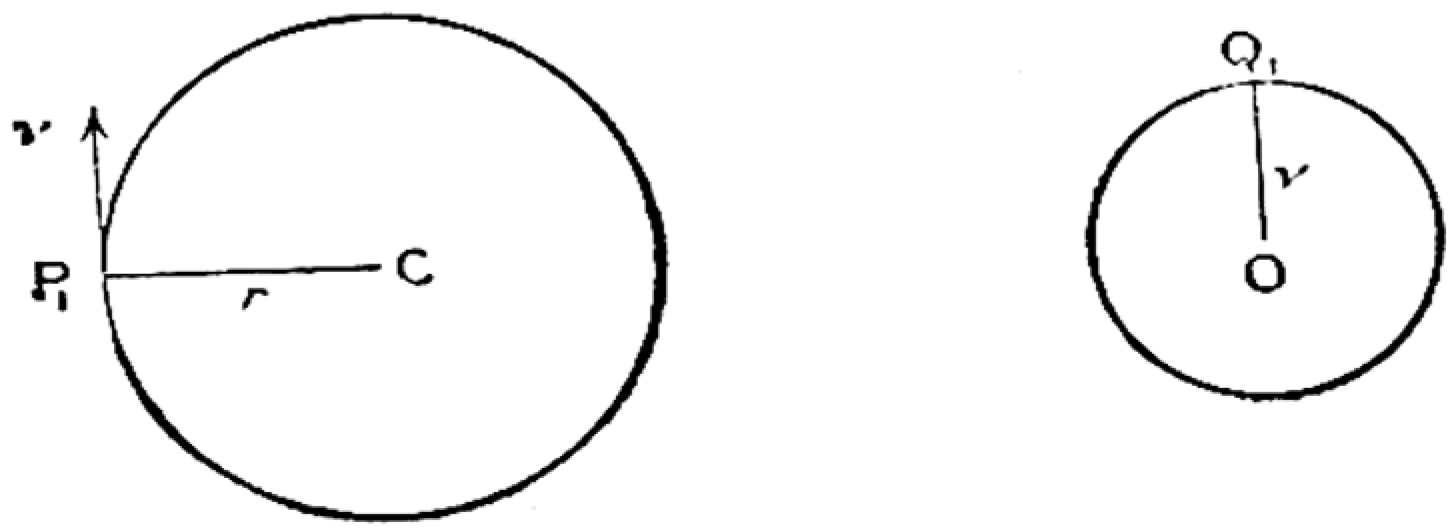

FIG. 4.

If P describes a circle of radius r with uniform speed v, the hodograph will be another circle of radius v, and in Fig. 4 Q_1

will be the point of the hodograph corresponding to P_1, OQ_1 being perpendicular to P_1C. Since P and Q describe their circles in the same time the velocity of $Q = v.\frac{v}{r}$ and its direction at Q_1 is perpendicular to OQ_1 and therefore parallel to P_1C. The acceleration of P is thus equal to $\frac{v^2}{r}$ and is directed towards C.

4. The Fictitious Centrifugal Force

Imagine a particle of mass m lying on a smooth horizontal surface and fastened by a light string of length r to a fixed point in the surface. If the particle is projected with a velocity v at right angles to the string, supposed just taut, it will describe a circle with constant speed v. The string will be in a state of tension and will exert a force on the particle directed towards the centre of the circle. (This is sometimes called the centripetal force.) It is this force which causes the motion in a circle and its magnitude must be $\frac{m.v^2}{r}$. The only other forces on the particle, its weight and the normal reaction of the surface, balance each other. The tension of the string also gives rise to a force on the fixed point to which it is attached. This force will be equal and opposite to the force on the particle. It is directed outwards from the centre and might appropriately be termed a centrifugal force. Note, however, that it acts on the fixed support at the centre and not on the particle of mass m.

If the string breaks the mass m will proceed along the tangent to the circle with constant velocity v.

As a further example, consider the toy engine of the nursery running round its circular track with constant speed. The spring is wound up and the back wheels have a strong tendency to rotate. They press backwards on the rails, the rails exert an equal and opposite force on the engine. If the engine is travelling at constant speed this force is just sufficient to balance the various resistances to the forward motion. The weight of the engine is balanced by the normal reaction of the rails. Also, owing to the tendency of the engine to proceed in a straight line, the flanges of the outer wheels are continually running into the outer rail of the circular track. The flanges exert a force outwards from the centre of the track on this rail, the rail exerts an equal and opposite force on the engine. This force is directed inwards and is the force which causes the circular motion of the engine. The force outwards on the rail is sometimes called the

centrifugal force. This is the centrifugal force of popular usage. The popular mind goes further and thinks of this force as somehow acting on the engine, and considers it to be the cause of the engine's toppling over if its speed is excessive. This confusion has risen out of a rather artificial method of treating rotation problems. Compare the two following treatments of the problem of the conical pendulum.

(1) The pendulum bob of mass m is describing a horizontal circle of radius r with uniform speed v, the light string fastened to a fixed point O making an angle θ with the vertical.

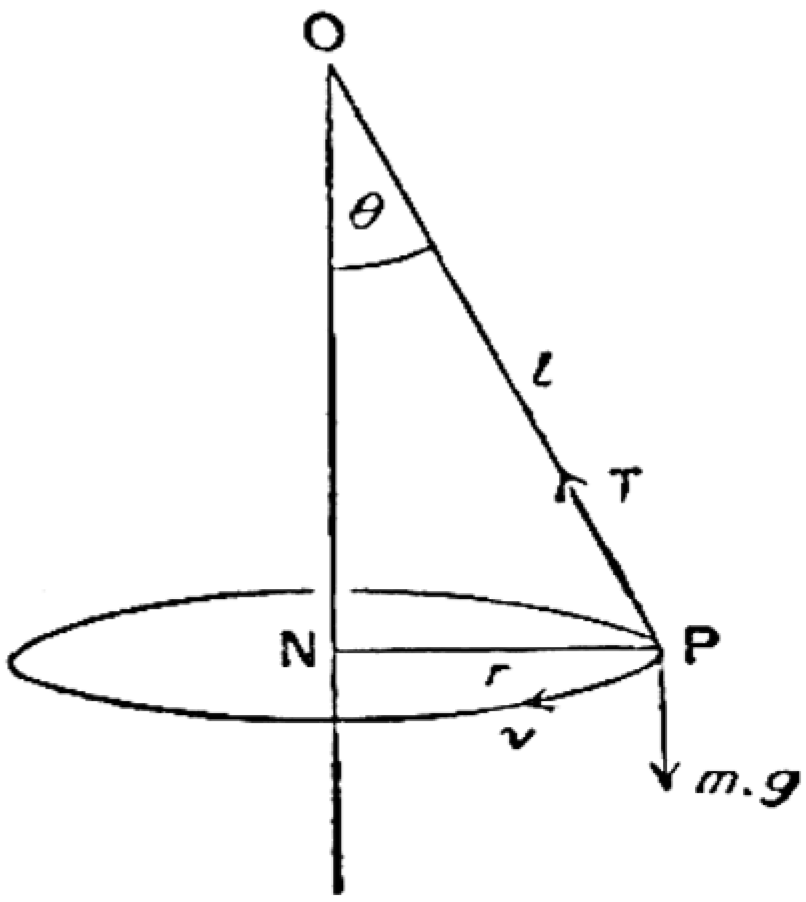

FIG. 5.

There are only two forces acting on the bob, its weight $m.g$ and T the tension of the string. The resultant of these, for any position of the bob in the circle, must be a force $\frac{m.v^2}{r}$ towards the centre of the circle. The resolved parts of T and $m.g$ along the radius must therefore be equal to $\frac{m.v^2}{r}$.

$$i.e., \text{T} \sin \theta = \frac{m.v^2}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

Also, since the mass m does not rise or fall, its acceleration in the vertical direction is zero.

Hence $$\text{T} \cos \theta = m.g. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

These two equations completely determine the motion.

(2) The second method consists in introducing a fictitious centrifugal force $\text{F} = \frac{m.v^2}{r}$ acting on the bob outwards from the centre. The problem is now considered to be one in Statics, the bob is assumed to be at rest and in equilibrium under the two real forces T and $m.g$ and the fictitious force F.

Resolving horizontally and vertically, we have again

$$\text{T} \sin \theta = \frac{m.v^2}{r}$$

and $$\text{T} \cos \theta = m.g,$$

or the Triangle of Forces may be applied.

Deductions from these equations are made in section 6.

This device, of the fictitious centrifugal force, is used by engineers, since it brings rotation problems within the scope of the methods of Graphical Statics. It is clearly convenient and requires justification because, actually, the particle is not in equilibrium. From a consideration of the motion of a particle relative to rotating axes, it may be shown that the introduction of the fictitious centrifugal force and the assumption of equilibrium will give correct results if the particle is rotating about an axis with constant angular velocity and if the distance of the particle from the axis remains constant. Most of the cases we meet satisfy these conditions. If these conditions are not satisfied it is necessary to introduce other fictitious forces in addition to $\frac{m \cdot v^2}{r}$ before we can ignore the rotation and treat the problem as one in Statics.

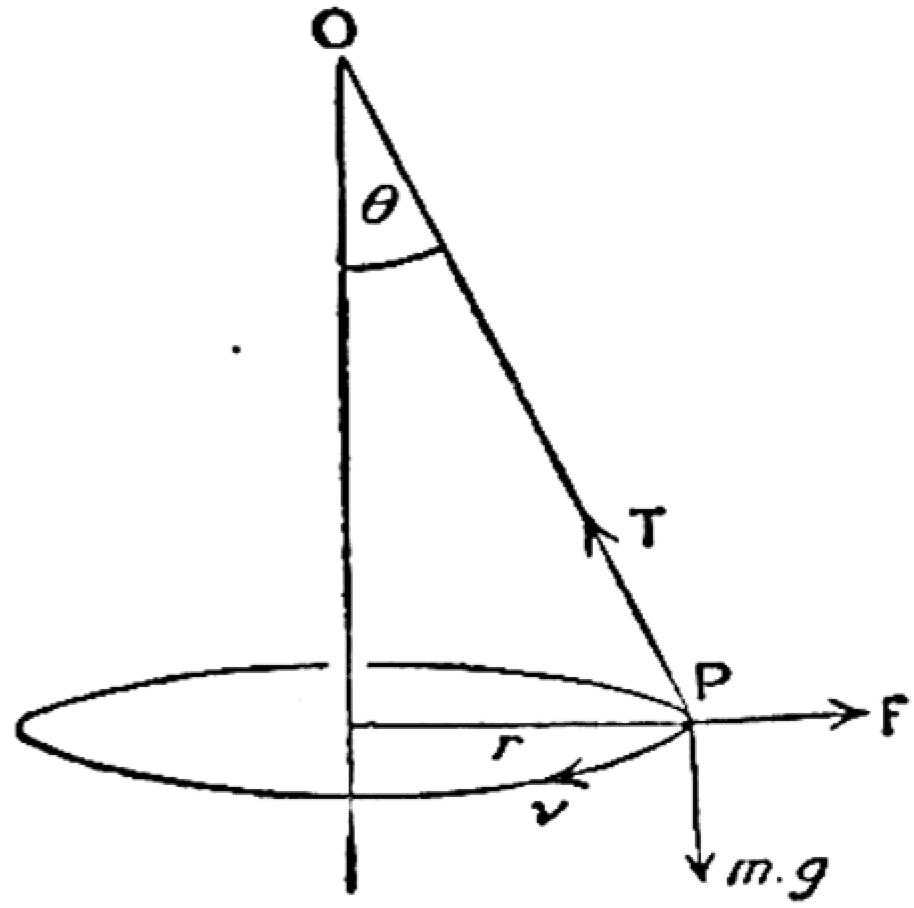

FIG. 6.

5. The Two Methods of Dealing with Rotation Problems

The first method uses the results of sections 1 and 2 of this chapter and Newton's Laws of Motion, viz. :—

(1) Every body continues in its state of rest or of uniform motion in a straight line, except in so far as it is caused to depart from this state by external impressed force.

(2) The rate of change of momentum of a body is proportional to the impressed force and is in the direction of the force.

(3) To every Action there is an equal and opposite Reaction.

In the second method the fictitious centrifugal force is introduced and the particle considered to be in equilibrium under the action of the real forces and the centrifugal force. If the body executing circular motion cannot be treated as a particle it is not clear at what point in the body the centrifugal force should be introduced. It is not correct in general to make the centrifugal force pass through the centre of mass of the body, although in special cases this point may lie on the line of action of the fictitious force. This difficulty is considered later.

The Conditions of Equilibrium of Coplanar Forces acting on a Body

(1) If there are only two forces they must be equal and opposite in direction and have the same line of action.

(2) If there are three forces their lines of action must meet in a point or be parallel. If the forces are parallel their algebraic sum must be zero and the algebraic sum of their moments about any one point in their plane must be zero.

If the forces are not parallel they must be capable of representation in magnitude and direction by the three sides of a triangle taken in order.

(3) If there are any number of forces three conditions are necessary and sufficient for equilibrium :—

(*a*) The algebraic sum of the resolved parts of the forces in some one direction must be zero.

(*b*) The algebraic sum of the resolved parts of the forces in some other direction (but not the opposite direction) must be zero.

(*c*) The algebraic sum of the moments of the forces about some one point in their plane must be zero.

Cases 1 and 2 are included in 3. A body acted on by coplanar forces will be in equilibrium if these conditions are satisfied. Note also that if a body is in equilibrium the algebraic sum of the components of the forces along any direction must be zero, and the algebraic sum of the moments of the forces about any point in their plane must also be zero.

6. The Conical Pendulum

In Fig. 5 the particle P of mass m describes a horizontal circle, centre N, radius r. The light string, of length l, is fastened to a fixed point O at a distance h vertically above N. Let the string make an angle θ with the vertical. If the speed of P is v its angular velocity $\omega = \frac{v}{r}$.

From section 4 we have,

$$\mathrm{T} \sin \theta = \frac{m \,.\, v^2}{r} = m \,.\, \omega^2 \,.\, r$$

and $$\mathrm{T} \cos \theta = m \,.\, g.$$

Squaring and adding, $\mathrm{T}^2 = m^2 \,(g^2 + \omega^4 \,.\, r^2)$

Dividing, $$\tan\theta = \frac{\omega^2 . r}{g} = \frac{r}{h} \text{ (from Fig. 5)}$$

$$\therefore\ h = \frac{g}{\omega^2} = l\cos\theta.$$

Also the time of revolution $= \dfrac{2\pi}{\omega} = 2\pi\sqrt{\dfrac{h}{g}}$. This is the same as the time of oscillation of a simple pendulum of length h.

The time of revolution may also be written $2\pi\sqrt{\dfrac{l\cos\theta}{g}}$.

Thin Rod

To illustrate the use of the centrifugal force method let us consider a thin rod free to rotate about one end in the manner of a conical pendulum. Each particle of the rod describes a horizontal circle with angular velocity ω. Let the length of the rod be l and its inclination to the vertical θ. The real forces on the rod are its weight M.g acting through its middle point, and the reaction R of the joint at O making an angle ϕ with the vertical. Let PQ be an element dx of the rod at a distance x

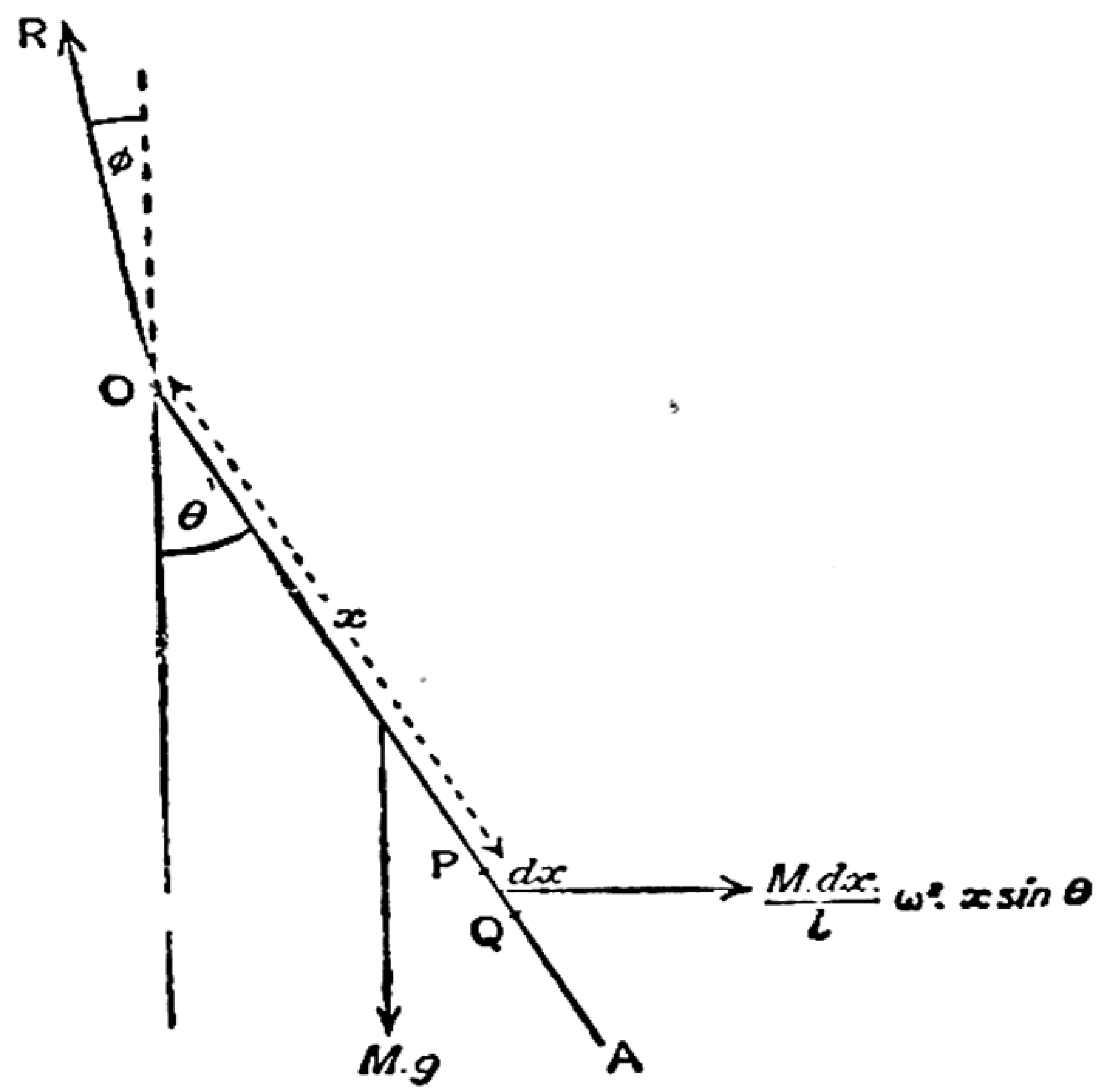

FIG. 7.

from O. The mass of this element is $\frac{M.dx}{l}$. We have to introduce a centrifugal force $\frac{M.dx}{l}.\omega^2.x\sin\theta$ acting on it, and similar centrifugal forces on all other elements of the rod. These forces are parallel and their resultant F is their sum. Thus

$$F = \int_0^l \frac{M.dx}{l}.\omega^2.x\sin\theta = \frac{M.\omega^2.\sin\theta}{l}\int_0 x\,dx$$

$$= \frac{M.\omega^2.\sin\theta}{l}.\frac{l^2}{2} \qquad = M.\omega^2.\frac{l}{2}.\sin\theta.$$

This resultant will act at a distance $\bar{x}$ from O such that

$$F.\bar{x}.\cos\theta = \int_0^l \frac{M.dx}{l}.\omega^2 x\sin\theta.x\cos\theta$$

$$= \frac{M.\omega^2.\sin\theta\cos\theta}{l}\int_0^l x^2\,dx$$

$$= \frac{M.\omega^2.\sin\theta\cos\theta}{l}.\frac{l^3}{3}.$$

$$\text{So } \bar{x} = \frac{l^3}{3} \div \frac{l^2}{2} = \frac{2}{3}.l\ .$$

Notice that the magnitude of F is the same as if the whole mass were concentrated at the middle point of the rod, but the point of application of F is not the centre of mass, but a point $\frac{2}{3}.l$ from O.

Taking moments about O we have

$$Mg.\frac{l}{2}.\sin\theta = F.\frac{2}{3}l\cos\theta = M.\omega^2.\frac{l}{2}.\sin\theta.\frac{2}{3}.l.\cos\theta$$

i.e., $$\frac{g}{\omega^2} = \frac{2}{3}l\cos\theta.$$

The time of revolution $$= \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{\frac{2}{3}l\cos\theta.}{g}}$$

The time of revolution is the same as that of a simple conical pendulum of $\frac{2}{3}$ the length of the rod.

Resolving vertically $\quad R\cos\phi = M.g.$

Resolving horizontally $R \sin \phi = F = M.\omega^2.\frac{l}{2}.\sin \theta.$

Whence
$$\tan \phi = \frac{\omega^2}{g}.\frac{l}{2}.\sin \theta$$
$$= \frac{3}{4} \tan \theta.$$

7. Banking

When a railway truck is proceeding round a bend the flanges of its outer wheels press against the outer rail, owing to the tendency of the truck to proceed in a straight line. This rail, of course, exerts an equal and opposite force on the truck. If the truck is moving round a circular bend with constant speed the reaction of the rails must provide the required force $\frac{m.v^2}{r}$ towards the centre of the bend. The stress between rail and flange would result in excessive wear, so the outer rail is raised above the inner and the horizontal component of the normal reaction of the rails supplies the whole, or a considerable part, of the necessary force towards the centre of the bend. The same principle is involved in the banking of turns on motor-racing tracks and in the flight of aeroplanes. To change its direction an aeroplane must bank. There is no other way of providing the necessary inward normal force.

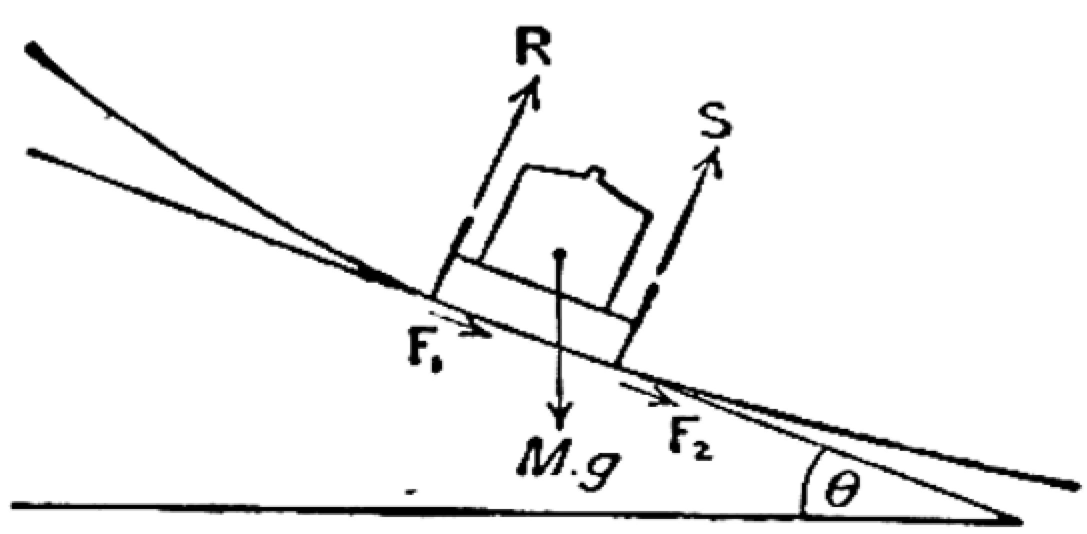

FIG. 8.

In Fig. 8 the forces shown are those acting on a car moving at speed v round a bend of radius r. If the speed falls below a certain value the forces F_1 and F_2 will be reversed in direction. The horizontal components of the forces in Fig. 8 must be equal to $\frac{M.v^2}{r}$ and the vertical components must add up to zero.

Thus $$(R + S) \sin \theta + (F_1 + F_2) \cos \theta = \frac{M.v^2}{r}$$

and $$(R + S) \cos \theta - (F_1 + F_2) \sin \theta = M.g.$$

To obtain the angle of banking just suited to the speed we put $F_1 + F_2 = 0$. The resultant reaction between ground and

wheels is now perpendicular to the ground, and the car runs as if on a level straight track with no side-strain on the tyres. The above equations become $(R + S) \sin \theta = \frac{M.v^2}{r}$

and $(R + S) \cos \theta = M.g,$

giving $\tan \theta = \frac{v^2}{g.r}$

and $v = \sqrt{g.r.\tan \theta.}$

If $v > \sqrt{g.r.\tan \theta}$ the horizontal component of $(R + S)$ will not be large enough to supply the necessary force towards the centre of the bend, and frictional forces F_1 and F_2 will be called into play. If $v < \sqrt{g.r.\tan \theta}$ F_1 and F_2 will act up the banking.

In the same way it may be shown that if the outer rail is raised, so that the track is tilted to an angle $\tan^{-1} \frac{v^2}{g.r}$ to the horizontal, the stress between flange and rail will vanish.

8. Tendency to Overturn or Skid

In Fig. 9 we have a vehicle describing a circular arc of radius r with speed v on level ground. Here it is convenient to use the centrifugal force. The line of action of the centrifugal force in this case passes through the centre of mass. This is because the axis of symmetry of the body is parallel to the axis of rotation.

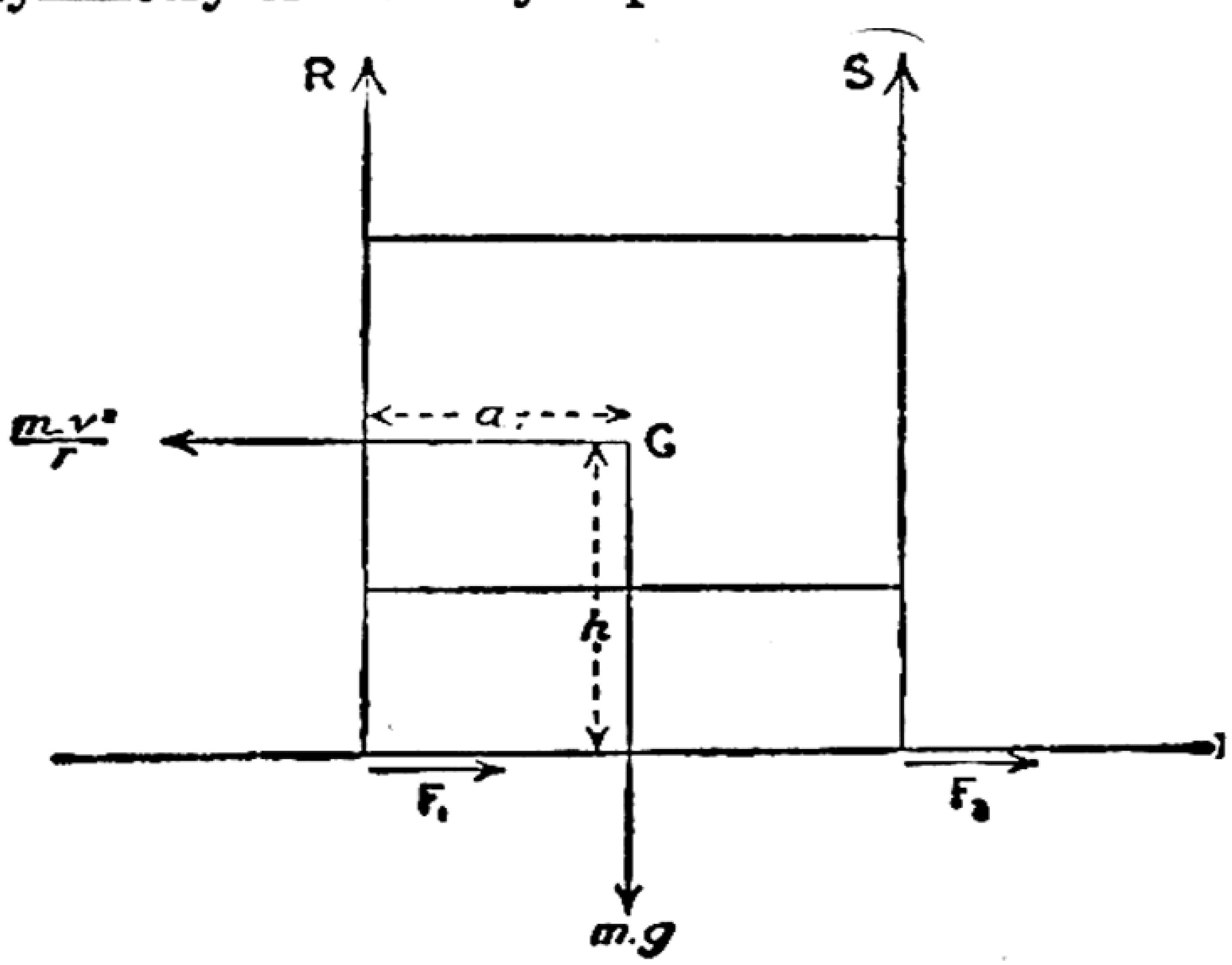

FIG. 9.

For a vehicle on banking the centrifugal force does not act through the centre of mass. The forces on the vehicle, together with the fictitious force, are shown on the diagram.

F_1 and F_2 are friction forces acting towards the centre of the arc. If the vehicle overturns it will turn about the points of contact of the outer wheels with the ground, and S and F_2 will become zero. It will clearly overturn if $\frac{m.v^2}{r}.h > mg.a$, where h is the height of the centre of mass above the ground and $2a$ is the distance between the wheels.

That is, if $v^2 > g.r.\frac{a}{h}$.

The vehicle will skid if

$$\frac{mv^2}{r} > \text{the maximum value of } (F_1 + F_2),$$

or
$$\frac{mv^2}{r} > \mu.mg,$$

or
$$v^2 > g.r.\mu,$$

where μ is the coefficient of friction between wheels and ground.

Thus if $\mu > \frac{a}{h}$ the vehicle will overturn rather than skid.

If $\mu < \frac{a}{h}$ it will skid before the speed is high enough to cause overturning.

If the vehicle is on a banked track and is describing a circular arc, the radius of the arc being large compared with the dimensions of the vehicle, the line of action for the centrifugal force passes very close to the centre of mass, and it is usual to assume that it passes through the centre of mass. With this assumption the conditions for overturning and skidding can be found as above. The student should satisfy himself that for a banking angle θ overturning occurs if $v^2 > g.r.\frac{h \tan\theta + a}{h - a\tan\theta}$; and skidding, if $v^2 > g.r.\frac{\mu}{\cos\theta}$.

9. Tension of a String in Circular Motion

A string in the shape of a circle of radius r revolves in its own plane about its centre O with speed v. The string will have a uniform tension T. If PQ is an element subtending an angle $\delta\theta$

at O its length is $r.\delta\theta$. The forces on the element are T at P and T at Q perpendicular to OP and OQ. Their resultant is the necessary normal force for the circular motion of PQ. So, if m is the mass of the string *per unit length*,

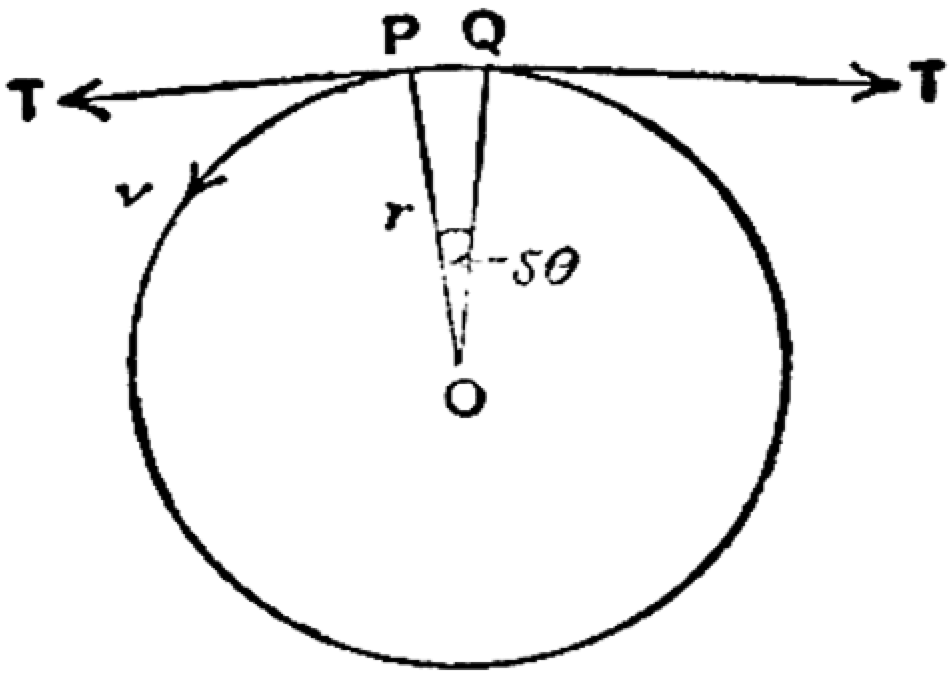

FIG. 10.

$$2T.\sin\frac{\delta\theta}{2} = m.r\delta\theta.\frac{v^2}{r}$$

$$\text{or } T.\delta\theta = mv^2.\delta\theta.$$

So $T = mv^2 = m\omega^2r^2$, where ω is the angular velocity.

If M is the total mass $T = \frac{M}{2\pi r}.\omega^2r^2 = \frac{M}{2\pi}.\omega^2r.$

The kinetic energy of the string $= \frac{1}{2}Mv^2 = \frac{1}{2}M\omega^2r^2 = \pi.r.T.$

EXAMPLES

1. Find the force required to keep a body of mass m gm. moving in a circular path of radius r cm. at a speed of v cm. per sec., and show that it acts through the centre of the circle.

Five passengers seat themselves on a 'joy-wheel' at distances of 1, 2, 3, 4 and 5 yards from its centre. If their coefficients of friction on the horizontal rotating platform are each ·2, find at what angular velocities sliding begins in each case. What would have to be their coefficients of friction for all of them to begin sliding together at an angular velocity of 1 radian per sec.?

2. A circular motion in a horizontal plane is described by a whirling body of mass 300 gm. attached to a string 50 cm. long whose breaking load is 4 kgm., calculate the maximum possible number of revolutions per minute. When the string breaks in what direction will the body move? (N.U.)

3. A particle starts from rest and moves in an arc of a circle with uniformly increasing speed. Show that, if ϕ is the angle between the direction of the force on the particle and the direction of the radius vector drawn to the particle, then $\phi = \tan^{-1}\frac{1}{2\theta}$; where θ is the angle through which the radius vector has turned since the start. (C. Schol.)

4. Explain why it is customary for the rails of a curved railway track to be of different heights.

A train travels round a curve of half a mile radius with a velocity of 40 m.p.h. If the distance between the rails is 5 ft., how much must the outer rail be raised above the inner rail in order that there may be no thrust on the flanges? If for this cant the speed is increased to 60 m.p.h., what will be the thrust on the flanges per ton weight of the train? (C.)

5. A motor van is steered to take a curve of radius 100 ft. in a horizontal plane. The wheel track of the van is 8 ft. wide, and its centre of gravity is 5 ft. above the ground and midway between the wheels. Find (*a*) the highest speed at which the van can take the curve without toppling over, and (*b*) the magnitude of the coefficient of limiting friction between the wheels and the ground in order that this speed may be maintained on the curve without skidding. (N.U.)

6. Explain what you understand by the term 'Centrifugal Force.' A circular motor track is half a mile in diameter, and a car is running round it at 60 m.p.h. Show what forces are acting on the car and find the angle of banking in order that there should be no tendency to side slip.

7. In a 'loop the loop' railway, the cars, after descending a steep incline, run round the inside of a vertical circular track, 20 ft. in diameter, making a complete turn over. Assuming there is no friction, find the minimum height above the top of the circular track from which the cars must start. (N.U.)

[If the required distance is h ft. the velocity of car at the top of the circular track is given by $v^2 = 2g.h$. The circular motion will be complete if at this point $\frac{mv^2}{10} > mg$. (The car must be pressing against the track) *i.e.*, $v^2 > 10g$, or $2gh > 10g$, or $h > 5$ ft. Minimum height is, therefore, 5 ft.]

8. A is the highest point of a fixed smooth sphere whose centre is O. A particle P, starting from rest at A, slides under the action of gravity down the outside of the sphere. Prove that it will leave the sphere when $3 \cos \theta = 2$, where θ is the angle AOP. (C. Schol.)

9. An aeroplane describes a vertical circle when looping. Find the radius of the greatest possible loop if the velocity of the aeroplane at the lowest point of its path is 120 m.p.h. (C. Schol.)

10. An india-rubber band has a mass of 4 gm. per metre when stretched on the circumference of a wheel 10 cm. radius, the stretching force being 20,000 dynes. Find how many revolutions per sec. the wheel must make so that the band may not press upon the wheel. (N.U.)

11. A string of length l is attached at one end to a fixed point P, there being a small body of mass m at the other end. The string is extended horizontally and the mass is then liberated. When the string has swung into the vertical position it encounters a peg Q distant a vertically below P, and the mass then performs a circular motion about Q. Calculate the tension in the string just before and just after the peg is encountered. (C. Schol.)

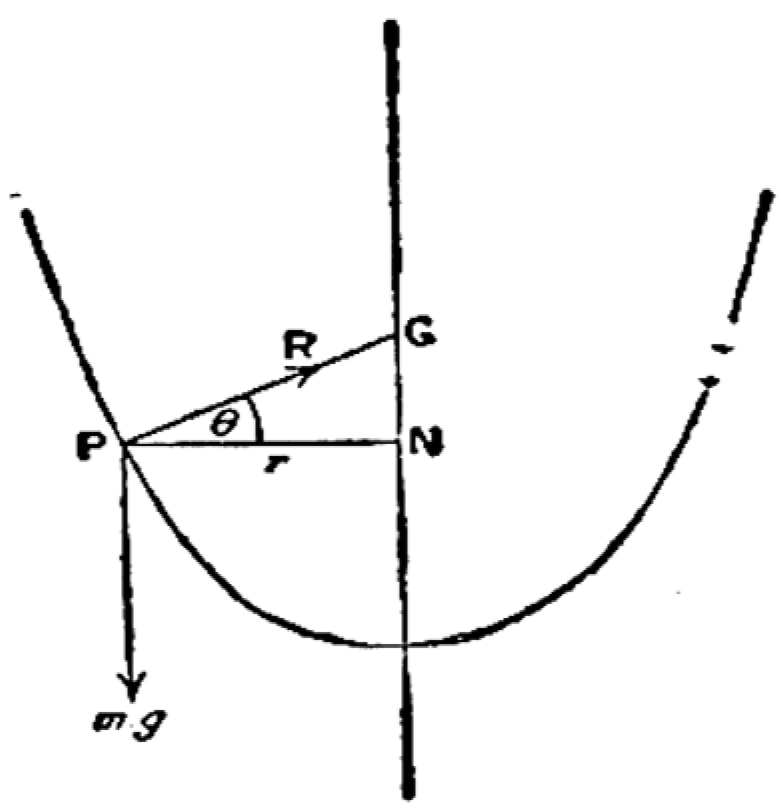

FIG. 11.

12. Deduce the shape of a hollow vessel such that, if it rotates about a vertical axis with angular velocity ω, a particle will remain in equilibrium at any point inside. (O. Schol.)

[The vessel must be smooth or the problem is indeterminate. Let P be the particle. The only forces on it are its weight mg and the normal reaction R. P describes a circle about N on the axis. Let the line of action of R meet the axis in G. Then $R \sin \theta = mg$ and $R \cos \theta = m.\omega^2.r$. Therefore $\tan \theta = \frac{g}{\omega^2 r}$, and $r \tan \theta = \frac{g}{\omega^2}$. But $NG = r \tan \theta = \frac{g}{\omega^2} =$ constant. The section of the vessel by any plane through the axis of revolution is a parabola whose latus rectum is $\frac{2g}{\omega^2}$, since the sub-normal $NG = \frac{g}{\omega^2}$ is constant. This is also the shape of the surface of a liquid contained in a vessel rotating about its vertical axis, as may be seen by considering the forces acting on a particle of the liquid in the surface. These are its weight and the resultant thrust from the underlying liquid. The latter must be normal to the surface, and the conditions are the same as above.]

13. A vertical cylinder filled with water rotates about its axis: find how the pressure in the water varies with the distance from the axis. If the water contained air bubbles, where would these collect? Explain the principle of the method of centrifugal separation. (C. Schol.)

[Take cylinders radii r and $r + \delta r$ having the same axis as the vessel. Consider an element, of area δA and thickness δr, of

the cylindrical shell so formed. Let the pressure in the liquid, on the same level as the element, at distance r from the axis be p, and $p + \delta p$ at distance $r + \delta r$. Then in a horizontal direction the force on this element is $(p + \delta p - p) . \delta A$ towards the axis. If the angular velocity is ω, and the density of the liquid ρ, we have $\delta p . \delta A = \rho . \delta A . \delta r . \omega^2 . r$.

Whence $\dfrac{dp}{dr} = \rho . \omega^2 . r$ and $p = \rho\omega^2 . \dfrac{r^2}{2} + C$, where C is the pressure in the same horizontal plane at the axis.

This pressure distribution will maintain particles of the liquid, or other particles of the same density, at rest relative to the body of the liquid. Particles of greater density than the liquid will require a larger inward force, in order to remain at relative rest, than is supplied by the pressure distribution. Their distance from the axis will steadily increase until they reach the wall of the containing cylinder. For particles less dense than the liquid the distribution of pressure provides a greater inward force than is necessary to maintain circular motion. Such particles will move towards the axis.]

14. A beaker containing water is fixed to the rim of a wheel which revolves in a horizontal plane with constant angular velocity. Calculate the inclination of the water surface to the horizontal, assuming that the surface may be considered to be plane. (C. Schol.)

CHAPTER II

SIMPLE HARMONIC MOTION

1. Simple Harmonic Motion (S.H.M.) as Orthogonal Projection of Uniform Circular Motion

CONSIDER a point P describing a circle, radius a, with uniform angular velocity ω. Let AOA′ be a diameter of the circle, centre O, and PN perpendicular to AOA′. Let the time be reckoned from the instant when P is at A. Then $\theta = \omega . t$. As

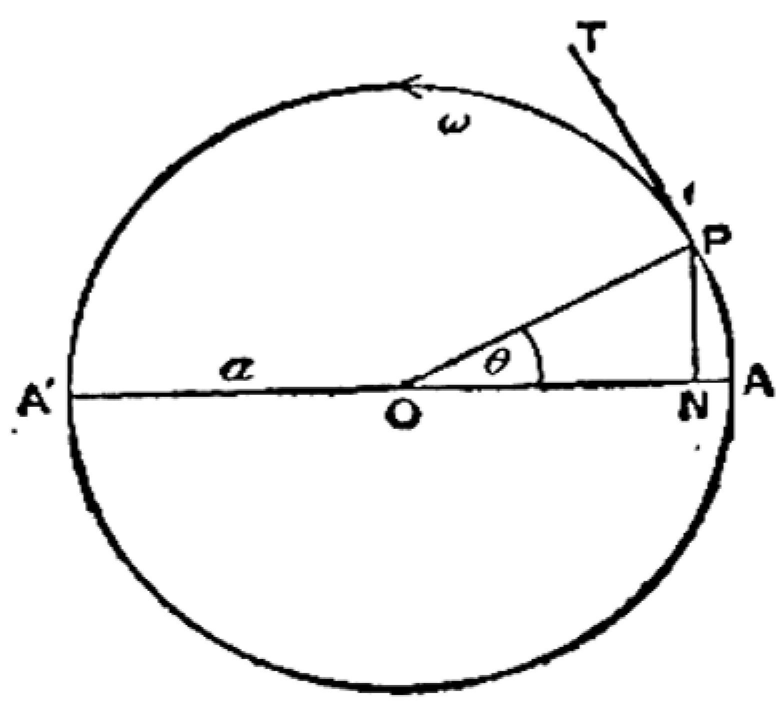

FIG. 12.

P describes the circle N moves from A through O to A′ and then back to A, and so on. The motion of N is that of P resolved parallel to AOA′.

The acceleration of P $= \omega^2 . a$ along PO.
Therefore the acceleration of N $= \omega^2 . a . \cos\theta$ along AO,
$= \omega^2 .$ ON towards O.
The velocity of P $= \omega . a$ along PT.
Therefore the velocity of N $= \omega . a . \sin\theta$ along AO,
$= \omega .$ PN along AO.

The acceleration of N is a maximum, $\omega^2 . a$, at A and A′, and zero at O. It is always proportional to the distance of N from O and directed towards O. Such motion is called Simple Harmonic. The distance $a = \text{OA} = \text{OA}'$ is called the amplitude of the motion.

The velocity of N is zero at A and A′, and a maximum, $\omega . a$, at O.

The period or periodic time of the motion of N is the interval between successive passages in the same direction through any fixed point in its path, and is equal to the time of revolution of P, viz., $\frac{2\pi}{\omega}$.

Notice also that the acceleration of N may be written $\omega^2 . a . \cos \omega t$, or if $ON = x$, $\omega^2 . x$; and that the velocity of N is $\omega . a \sin \omega t$, or $\omega\sqrt{a^2 - x^2}$.

2. Definition and Importance of S.H.M.

A particle is describing a S.H.M. if its acceleration is proportional to its distance, measured along its path, from a fixed point in its path and is directed towards that point.

If the particle is moving along the x axis, its co-ordinate being x at time t, its acceleration is $\frac{d^2x}{dt^2}$. If $\frac{d^2x}{dt^2} = -\mu x$, μ being a constant, then the acceleration is proportional to the distance of the particle from the origin and, owing to the minus sign, is directed towards the origin. This, then, is the mathematical form of the definition of S.H.M. By integrating this equation the velocity and displacement of the particle at any time may be found when the velocity and displacement at some particular time are given.

In section 1 the acceleration of N was found to be $\omega^2 . ON$ towards O, or $-\omega^2 . x$ if O is the origin and $ON = x$. The period was $\frac{2\pi}{\omega}$, hence the period of the S.H.M. $\frac{d^2x}{dt^2} = -\mu x$ is $\frac{2\pi}{\sqrt{\mu}}$. It is independent of the amplitude. Also the velocity when the displacement is x will be $\sqrt{\mu}\ \sqrt{a^2 - x^2}$, where a is the amplitude of the motion.

To integrate $\ddot{x} = -\mu x$ multiply through by $2\dot{x}$.

We have $2\dot{x} . \ddot{x} = -\mu x . 2\dot{x}$.

Integrating, $(\dot{x})^2 = -\mu x^2 + C$.

If $x = a$ when $\dot{x} = 0$, then $C = \mu a^2$, and $\dot{x}^2 = \mu(a^2 - x^2)$; a is the amplitude.

Therefore $\dot{x} = \sqrt{\mu}\ \sqrt{a^2 - x^2}$, giving velocity in terms of displacement,

or
$$\frac{dx}{\sqrt{a^2 - x^2}} = \sqrt{\mu} . dt.$$

Integrating, $\sin^{-1}\frac{x}{a} = \sqrt{\mu}.t + \alpha$, where α is a constant whose value depends on the position of the particle when $t = 0$.

We have, then, $x = a\sin(\sqrt{\mu}.t + \alpha)$ . . . . (1)

If $\quad x = a$ when $t = 0$, $\alpha = \frac{\pi}{2}$

and $\quad x = a\cos\sqrt{\mu}.t$.

If $\quad x = 0$ when $t = 0$, $\alpha = 0$

and $\quad x = a\sin\sqrt{\mu}.t$.

Equation (1) is the general equation giving the displacement at time t. Differentiating this, we have $\dot{x} = a\sqrt{\mu}\cos(\sqrt{\mu}.t + \alpha)$, giving the velocity at time t.

S.H.M. owes its importance to the fact that most oscillations of small amplitude are of this type. It plays a great part in the dynamical treatment of Sound and in the theory of Alternating Current. Periodic motions, *i.e.*, motions which after a certain time repeat themselves are not in general simple harmonic, but they can be analysed into a series of simple harmonic motions.

The motion of the cross-head in a steam engine is periodic. It is approximately S.H.M. if the connecting rod is long compared to the crank arm.

The motion of the bob of a simple pendulum is S.H. for small amplitudes. If a second pendulum is suspended from the bob of the first and allowed to oscillate through a small angle in a vertical plane, the motions of both bobs will be periodic, but not necessarily S.H.

3. Force and Energy in S.H.M.

If the resultant force F, acting on a particle of mass m moving in a straight line, is always proportional to the distance of the particle from a fixed point in its path and is directed towards that point, the motion of the particle is S.H. If we take the x axis as the line of motion and the fixed point as the origin, then $F = -k.x$. Therefore $m\ddot{x} = -kx$ and $\ddot{x} = -\frac{k}{m}.x$. The period of the motion is $2\pi\sqrt{\frac{m}{k}}$. Notice that k is the strength of the force when the displacement is one unit. The force F acting towards the fixed point and proportional to the displacement is called the restoring force.

(The dot notation is Newton's ; $\dot{x} = \frac{dx}{dt}$, $\ddot{x} = \frac{d^2x}{dt^2}$.)

Consider the S.H.M. defined by $\ddot{x} = -\mu x$, and suppose the amplitude to be a, the mass of the particle being m. When the displacement is x, the velocity $\dot{x} = \sqrt{\mu}\sqrt{a^2 - x^2}$; and the Kinetic Energy $= \frac{1}{2}m\dot{x}^2 = \frac{1}{2}m.\mu\,(a^2 - x^2)$.

The force $F = m\ddot{x} = -m.\mu x$, and the Potential Energy $= -\int_0^x F.dx = \frac{1}{2}m.\mu x^2$.

$$\text{The Total Energy} = \tfrac{1}{2}m.\mu\,(a^2 - x^2) + \tfrac{1}{2}m.\mu x^2 = \tfrac{1}{2}m.\mu.a^2.$$

Notice that the Total Energy is constant and proportional to the square of the amplitude. In the above work Force and Energy are measured in absolute units, either c.g.s. or lb. ft. sec.

4. The Simple Pendulum

A small sphere A of mass m hangs from B by a length l of thread or fine wire whose mass is neglected. It is allowed to swing in a vertical plane through a small angle 2θ. It swings along an arc about a mean position O. This arc may be considered a horizontal straight line if θ is small. The forces on the sphere A are as shown in Fig. 13.

If θ is small $T = mg$, and the restoring force on A towards O is $T \sin\theta = mg \sin\theta = mg.\frac{x}{l}$.

So
$$m\ddot{x} = -mg.\frac{x}{l},$$
or
$$\ddot{x} = -\frac{g}{l}.x.$$

The motion is, therefore, simple harmonic, and the period is $2\pi\sqrt{\frac{l}{g}}$.

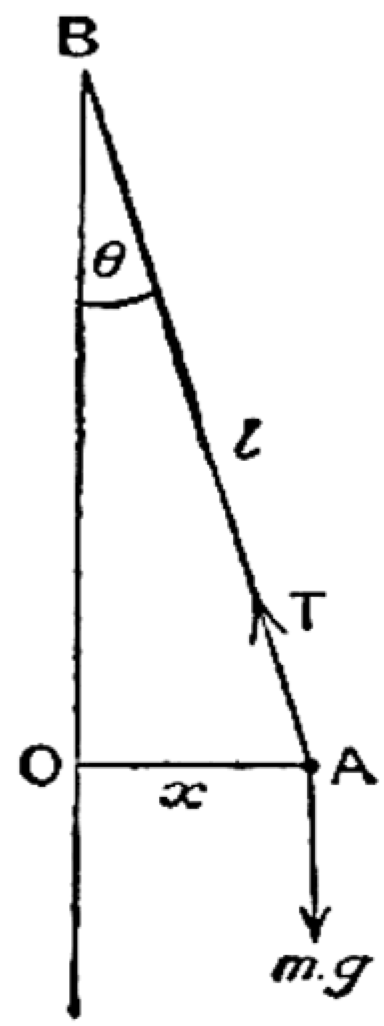

FIG. 13.

In the above l is the length of the string and the radius of the sphere is supposed to be extremely small. In practice, when the simple pendulum is used to measure g the sphere will have a radius which is not negligible. The corrections to the above simple formula for the period necessary to obtain an accurate value of g are discussed in Chapter IV.

5. Dimensions of Units

The fundamental units used in Mechanics are those of Mass,

Length and Time. In the c.g.s. system they are the gramme, centimetre and second; in the British system, the pound, foot and second. Other units, such as those of force, velocity, energy, are called derived units.

In numerical calculations it is generally advisable to express the data in terms of the fundamental units, and it is essential that in any one equation all the quantities involved shall be in the same units. For example, if a velocity is expressed in feet per second, a distance in the same equation must not be in inches.

The unit of velocity involves the fundamental units of length and time, and its magnitude depends on the units chosen for length and time being proportional to the unit of length and inversely proportional to that of time. It is said to have dimensions, one in length and minus one in time. This is expressed by the dimensional equation $V = L^1T^{-1}$.

The unit of acceleration varies as the unit of length and inversely as the square of the time unit, *i.e.*, $A = L^1T^{-2}$.

Force, being the product of mass and acceleration, has dimensions $M^1L^1T^{-2}$.

Work and Potential Energy are the product of force and distance and have dimensions $M^1L^1T^{-2}.L^1$ or $M^1L^2T^{-2}$.

Kinetic Energy is the product of mass and the square of velocity and has dimensions $M^1(L^1T^{-1})^2$ or $M^1L^2T^{-2}$. The dimensions of a constant, $\frac{1}{2}$ in this case, are zero in mass, length and time.

The dimensions of angular velocity are T^{-1}, since those of an angle (length of arc divided by length of radius) are zero.

This idea of dimensions is useful to keep in mind. It serves often as a valuable check on accuracy, owing to the fact that in any general equation in Mechanics and Physics the dimensions of every term must be the same. Notice above that the dimensions of Work, Potential Energy and Kinetic Energy are identical. Or take the equation $P.t = mv - mv_o$, expressing the fact that the change in momentum produced by a force is equal to the impulse of the force. The dimensions of Impulse, force × time, are $M^1L^1T^{-2}.T$ or $M^1L^1T^{-1}$; those of Momentum, mass × velocity, $M^1L^1T^{-1}$.

It is often possible to make use of this fact to determine the functional relation between one physical quantity and others on which it depends. For example, the time of oscillation t of a simple pendulum has dimensions T^1. Now this time of oscillation t might depend on l the length of the pendulum, on m the

mass of the bob, and on g the acceleration due to gravity at the place.

Suppose $t \propto m^x l^y g^z$.

The dimensions of $m^x l^y g^z$ are $M^x L^y L^z T^{-2z}$, or $M^x L^{y+z} T^{-2z}$. But the two sides of the equation must have the same dimensions, so $-2z = 1, \quad y + z = 0 \quad \text{and } x = 0.$

Whence $z = -\frac{1}{2}, \quad y = +\frac{1}{2} \text{ and } x = 0.$

So $t \propto l^{\frac{1}{2}} g^{-\frac{1}{2}}$ or $\propto \sqrt{\frac{l}{g}}$.

Table of Dimensions

Mass	M	Impulse	MLT^{-1}
Length	L	Couple	ML^2T^{-2}
Time	T	Pressure	$ML^{-1}T^{-2}$
Velocity	LT^{-1}	Stress	$ML^{-1}T^{-2}$
Acceleration	LT^{-2}	Modulus of Elasticity	$ML^{-1}T^{-2}$
Density	ML^{-3}	Moment of Inertia	ML^2
Force	MLT^{-2}	Angular Velocity	T^{-1}
Work	ML^2T^{-2}	Angular Momentum	ML^2T^{-1}
Energy	ML^2T^{-2}	Surface Tension	MT^{-2}
Power	ML^2T^{-3}	Surface Energy	MT^{-2}
Momentum	MLT^{-1}		

6. Examples

1. The vibrations of a tuning fork are simple harmonic. Suppose the frequency is 512 vibrations per sec. Then the period $T = \frac{1}{512}$ sec. If the amplitude at the end is 1 mm., what is the maximum velocity and the maximum acceleration?

Let the motion of one end of the fork be represented by

$$\ddot{x} = -k.x,$$

then
$$T = \frac{2\pi}{\sqrt{k}} = \frac{1}{512} \text{ sec.}$$

So
$$k = 4\pi^2 \times 512^2.$$

The velocity $v = \sqrt{k}\,\sqrt{a^2 - x^2}$, and the maximum velocity

$$= \sqrt{k}.a$$
$$= 2\pi \times 512 \times \cdot 1 = 322 \text{ cm. per sec.}$$

The maximum acceleration
$$= k.a$$
$$= 4\pi^2 \times 512^2 \times \cdot 1$$
$$= 1\cdot 04 \times 10^6 \text{ cm. per sec. per sec.}$$

2. In Fig. 14 AB represents a spiral spring whose mass is small compared with the attached mass m. Let the unstretched length of the spring be a, and let the extension produced by m be l. This extension is proportional to the weight mg for moderate extensions,

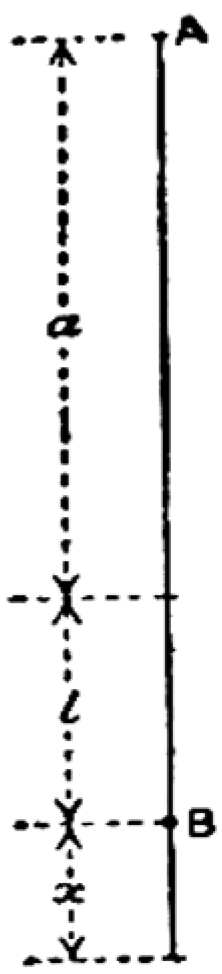

FIG. 14.

$$i.e., \ mg = \frac{\lambda}{a}.l,$$

where the constant λ is the force required to produce an extension equal to the original length. The force exerted by the spring upwards is equal now to mg and the mass m is in equilibrium. Suppose m is given a further displacement x and then released. The resultant force on m is now upwards and equal to

$$\frac{\lambda}{a}(l + x) - mg = \frac{\lambda}{a}.x.$$

Therefore $$m\ddot{x} = -\frac{\lambda}{a}.x,$$

since x increases downwards and the force is upwards.

The motion of m is thus S.H. about the equilibrium position, and the period

$$\mathrm{T} = 2\pi\sqrt{\frac{ma}{\lambda}} = 2\pi\sqrt{\frac{l}{g}},$$

and is independent of x.

It is the same as the period of a simple pendulum of length equal to the extension produced by m.

3. An elastic string of unstretched length a is stretched to a length $a + l$ between the fixed points A and B. At its middle

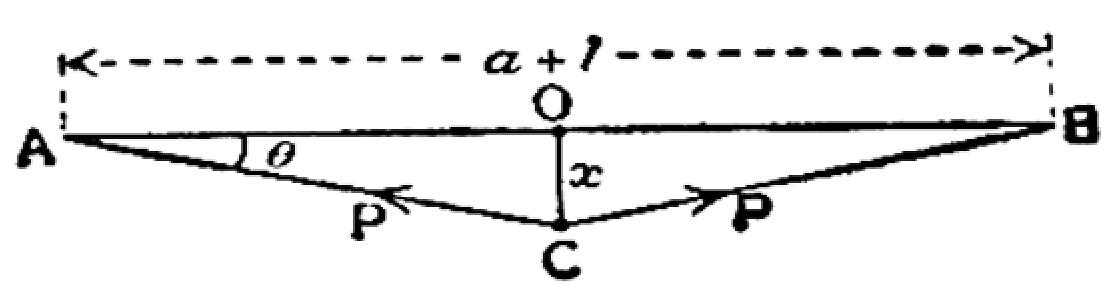

FIG. 15.

point is attached a mass m. The mass of the string is neglected. The whole rests on a smooth horizontal table. m is drawn to one side through a small distance x and released. Find the period of the ensuing vibration.

Let P be the tension of the string.

If $\left(\frac{x}{a+l}\right)^2$ be negligible, then ACB (Fig. 15) may be taken equal to $a + l$.

Thus $P = \frac{\lambda}{a} . l.$

The force on m towards $O = 2P \sin \theta = 2P . \frac{2x}{a+l} = \frac{4P . x}{a+l}$.

Therefore $$m\ddot{x} = -\frac{4P}{a+l} . x.$$

The motion is thus S.H. and the period

$$T = 2\pi\sqrt{\frac{m(a+l)}{4P}}$$

$$= \pi\sqrt{\frac{ma(a+l)}{\lambda . l}}$$

4. Show that two S.H.M.s of equal amplitude and period may, under certain circumstances, compound into uniform motion in a circle.

Let a point P have a S.H.M. of amplitude a and period $\frac{2\pi}{\sqrt{\mu}}$ about a point O along a line parallel to the x axis, and at the same time let the point O move up and down the y axis with a S.H.M. of the same amplitude and period about the origin as centre.

At time t the co-ordinates of P will be

$$x = a \cos(\sqrt{\mu} . t + \alpha),\ y = a \cos(\sqrt{\mu} . t + \beta),$$

where α and β are determined by the position of P when $t = 0$

If $$(\sqrt{\mu} . t + \alpha) = 90° + (\sqrt{\mu} . t + \beta),$$

then $$\cos(\sqrt{\mu} . t + \beta) = \sin(\sqrt{\mu} . t + \alpha)$$

and $$x^2 + y^2 = a^2.$$

The locus of P is thus a circle of radius a and centre at the origin. The two S.H.M.s compound into a circular motion if they are along two lines at right angles and if $\alpha - \beta = 90°$.

The geometrical meaning of the angles α and β is shown in Fig. 16. If the time is measured from the instant when P is at C, then after time t, ON $= a \cos(\sqrt{\mu} . t + \alpha)$.

Angle AOP is called the phase and α the initial phase. In the

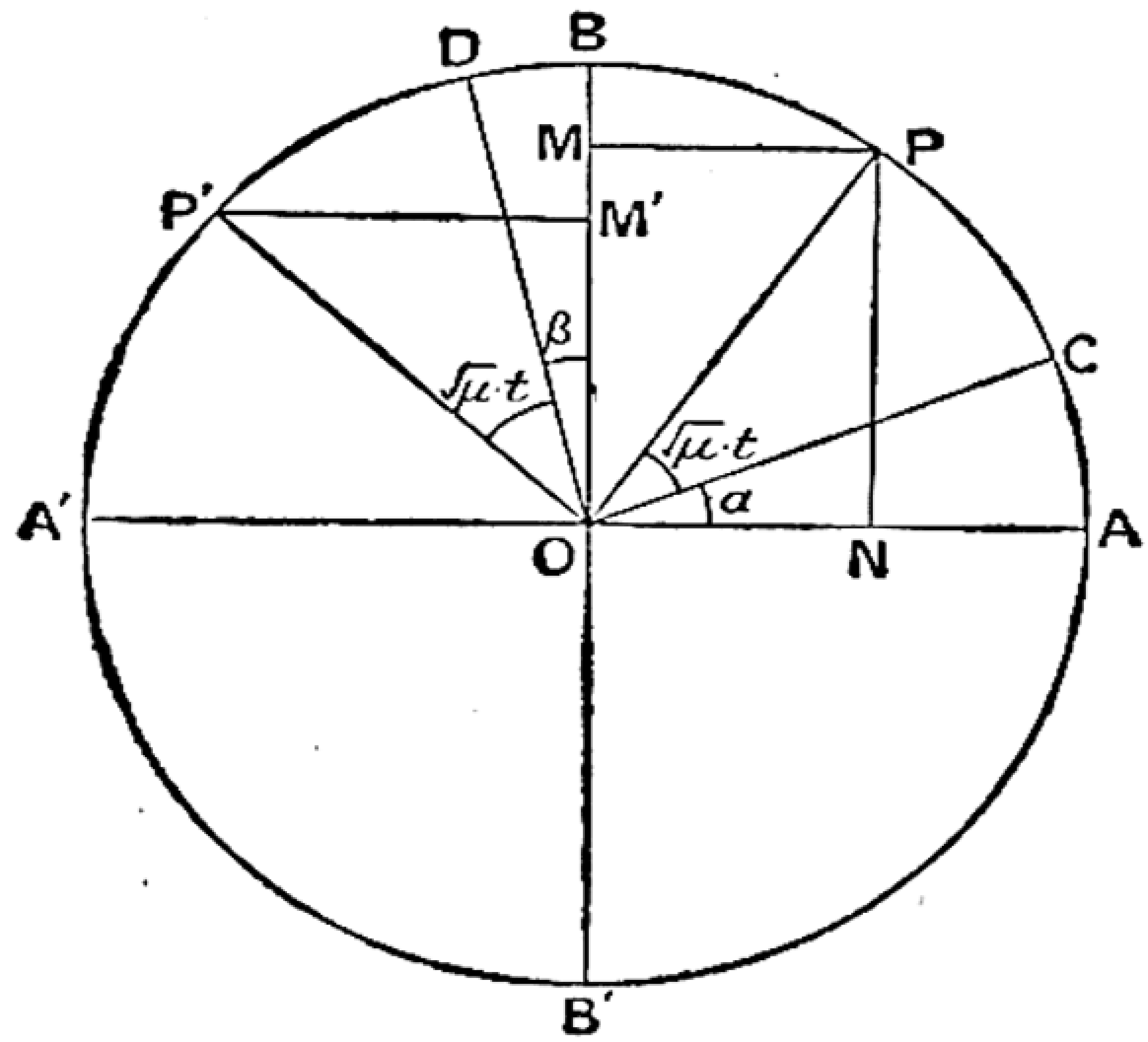

FIG. 16.

same way angle BOD is β the initial phase of the motion along BOB′. If $\beta = \alpha - 90°$ OP′ coincides with OP and the two S.H.M.s along AOA′ and BOB′ compound into the circular motion of P with angular velocity $\sqrt{\mu}$.

If $\alpha = \beta$ the two S.H.M.s compound into a S.H.M. along $y = x$ of the same amplitude and period.

EXAMPLES

1. A particle moves with uniform speed round the circumference of a circle. Show that its projection on any straight line in the same plane as the circle moves in such a way that its acceleration towards a fixed point is proportional to its displacement from that point.

A body performs S.H.M. with an amplitude of 10 cm. and a period of 5 secs. Find its velocity at the moment when its displacement is 5 cm. from the middle of its swing. (N.U.)

2. A particle describing S.H.M. starts at a distance of 5 ft. from a point A, and when its distance from A is 3 ft. its velocity is 20 f.p.s. Find its velocity when it passes through A. (O. and C.)

3. A hydrometer floats upright in a liquid, its displacement is 30 c.c. and the diameter of its stem ·8 cm. Prove that the time of a small vertical oscillation is 1·55 secs.

4. What is the most important characteristic of S.H.M., and why is this type of motion of much importance in Physics ?

A bottle weighing a kilogram is floated in a liquid whose specific gravity is 0·8. The cross-sectional area of the bottle about the level of the liquid is 5 sq. cm. Show that its oscillations are S.H. when it is disturbed and find their period. (C. Schol.)

5. Explain what is meant by S.H.M. and find the period of a body executing a linear S.H.M. in terms of its acceleration and displacement. A uniform U-tube contains a liquid to a height of 20 cm. If the liquid is slightly depressed and then released, show the motion is S.H. and find its period. (O. and C.)

6. Explain what is meant by S.H.M., and show how the velocity of a particle executing a linear S.H.M. varies with its displacement from the equilibrium position.

On suspending a mass of 3 lb. from the end of a spring of negligible mass there is an extension of 2 in. If the mass is slowly depressed a further distance of 1 in. and then released, find the periodic time of the oscillation and the maximum velocity of the particle. (C.)

7. A helical spring extends d cm. for every additional gram load. It is caused to vibrate when carrying a mass of m grams. Show the period of vibration is $2\pi\sqrt{\frac{d.m}{g}}$ (C. Schol.)

8. Prove that the motion of a simple pendulum, when of small amplitude, is approximately S.H. and find its period.

A clock pendulum which beats seconds on the earth's surface, where $g = 32$, is taken down a mine of depth $\frac{1}{4}$ mile. Assuming g to vary directly as the distance from the earth's centre, and that the earth's radius is 4,000 miles, find the number of seconds lost by the clock in a day. (O. and C.)

9. What would be the times of swing of a seconds pendulum vibrating in a lift (1) when the lift has a downward acceleration of $g/10$; (2) when it has an upward acceleration of the same amount ? (C. Schol.)

10. A body is moving with S.H.M. Find expressions for its potential and kinetic energies at any moment, and show that their sum is constant.

11. The bob of a simple pendulum of period 2 secs. has a maximum speed of 1 ft. per sec. Find the amplitude of the swing in degrees.

Discuss, without accurate computation, the effect produced

upon the angular amplitude if the length of the pendulum is slightly increased, the maximum speed remaining the same as before. (N.U.)

12. A and B start together at a point P on the circumference of a circle, radius a. A performs a S.H.M. along the diameter through P, whilst B describes the circle with uniform speed v. If the time of revolution of the latter is equal to the period of vibration of the former, discuss the motion of B relative to A and find an expression for the relative velocity at any instant. (C. Schol.)

CHAPTER III

ROTATIONAL MOTION OF RIGID BODIES

1. Kinetic Energy and Angular Momentum of a Rotating Body

A BODY is rigid if the forces acting on it produce no change in the distances apart of its component particles.

Consider a rigid body rotating about a fixed axis AB with angular velocity ω. The body is made up of a large number of particles of masses m_1, m_2, etc. Let r_1 be the perpendicular distance of m_1 from AB.

The velocity of $m_1 = r_1 . \omega$.
Its Kinetic Energy $= \frac{1}{2} m_1 r_1^2 \omega^2$.
Therefore Total Kinetic Energy $= \frac{1}{2}\omega^2 . \Sigma m_1 r_1^2$.
The momentum of $m_1 = m_1 r_1 \omega$.

The moment of momentum of m_1 about the axis AB $= m_1 r_1^2 . \omega$.
The Total Moment of Momentum about the axis $= \omega . \Sigma m_1 r_1^2$.

Moment of Momentum is sometimes called Angular Momentum.

2. Moment of Inertia and Radius of Gyration

The quantity $\Sigma m_1 r_1^2$ occurs in the expressions for the Kinetic Energy and Angular Momentum. It is called the Moment of Inertia of the body about the axis AB, and is denoted by the symbol I. It can be calculated for regular bodies, for others it has to be found by experiment.

Let $M = \Sigma m_1$, so that M is the total mass of the body. The length k such that

$$Mk^2 = I = \Sigma m_1 r_1^2,$$

is called the radius of gyration of the body about the axis AB.

The formulæ of section 1 may be written

$$\text{Kinetic Energy} = \tfrac{1}{2} I \omega^2 = \tfrac{1}{2} M k^2 \omega^2.$$
$$\text{Angular Momentum} = I\omega = Mk^2\omega.$$

The kinetic energy is, of course, given by this formula in absolute units.

3. Correspondence between Linear and Rotational Motion

The formulæ for the Kinetic Energy and Angular Momentum

of a body rotating about a fixed axis have the same form as the formulæ for the Kinetic Energy and Momentum of a mass m moving with velocity v, viz., $\frac{1}{2}mv^2$, mv. I takes the place of m, and ω of v. There is, indeed, a close parallel between the quantities involved in linear motion and corresponding quantities in rotation. The following table shows the correspondence :—

Linear Motion.		Rotational Motion.	
Mass	m	Moment of Inertia	I
Linear Velocity	v or $\frac{ds}{dt}$ or $\dot{s}$	Angular Velocity	ω or $\frac{d\theta}{dt}$ or $\dot{\theta}$
Linear Acceleration	a or $\frac{d^2s}{dt^2}$ or $\ddot{s}$	Angular Acceleration	$\dot{\omega}$ or $\frac{d^2\theta}{dt^2}$ or $\ddot{\theta}$
Force	$F = m.a$	Moment of Couple	$C = I.\ddot{\theta}$
Impulse	$F.t = m(v - v_0)$	Impulse of Couple	$C.t = I(\omega - \omega_0)$
Work	$F.s = \frac{1}{2}m(v^2 - v_0^2)$	Work	$C.\theta = \frac{1}{2}I(\omega^2 - \omega_0^2)$
Kinetic Energy	$\frac{1}{2}m.v^2$	Kinetic Energy	$\frac{1}{2}I.\omega^2$
Linear Momentum	$m.v$	Angular Momentum	$I.\omega$

Notice that in rotational motion the moment of a couple or moment of a force takes the place of force. The work done by a couple in turning a body through θ radians is $C \times \theta$, where C is the moment of the couple.

θ, ω and $\dot{\omega}$ are in terms of radians.

v_0, ω_0 are the initial values of v, ω.

The work done by a couple causing rotation is equal to the kinetic energy gained. The Impulse of the couple is equal to the increase in the Angular Momentum.

To illustrate this correspondence let us consider the simple pendulum. The pendulum rotates about an axis through B perpendicular to the plane of the paper (Fig. 13). The rotation is caused by the moment of the weight about this axis. The tension of the string and the reaction of the support have no moment about B. The moment of the weight is $mg.l.\sin\theta$ and is in the direction of θ decreasing. We have then

$$I.\ddot{\theta} = -mg.l.\sin\theta$$
$$= -mg.l.\theta, \text{ if } \theta \text{ is small.}$$

But $$I = m.l^2, \text{ so } \ddot{\theta} = -\frac{g}{l}.\theta.$$

The form of this equation indicates S.H.M. and the period $T = 2\pi\sqrt{\frac{l}{g}}$.

The massive particle and the light string (or light rod) are here treated as a rigid body.

4. Calculation of Moments of Inertia

The Moment of Inertia of a body about an axis is, as we have seen, Σmr^2. It involves the mass of the body, its linear dimensions, and the distance of the centre of mass of the body from the axis. The calculation for simple bodies requires an elementary knowledge of the Integral Calculus, and is greatly assisted by the two theorems of the next section.

(1) *Moment of Inertia of a thin rod about a perpendicular axis through its middle point.*

Let O be the middle point. Let M be the mass of the rod, a its length, and m its mass per unit length. Then $\mathrm{M} = a.m$. Let PQ be an element of the rod of length δx at a distance x from O. The mass of PQ is $m.\delta x$, and its M.I. $= m.\delta x.x^2$.

M.I. for the whole rod $= \Sigma mx^2.\delta x$

$$= m\int_{-\frac{a}{2}}^{+\frac{a}{2}} x^2.dx = m\left[\frac{x^3}{3}\right]_{-\frac{a}{2}}^{+\frac{a}{2}} = \frac{m}{3}\left\{\frac{a^3}{8} + \frac{a^3}{8}\right\}$$

$$= \frac{ma^3}{12} = \mathrm{M}.\frac{a^2}{12}.$$

The M.I. about an axis through one end of the rod will be

$$\int_o^a mx^2dx = m\left[\frac{x^3}{3}\right]_o^a = \frac{ma^3}{3} = \mathrm{M}.\frac{a^2}{3}.$$

(2) *Moment of Inertia of a circular thin ring about an axis through the centre perpendicular to the plane of the ring.*

Let a be the radius of the ring and M its mass. Since every bit of the ring is at the same perpendicular distance from the axis, the M.I. $= \Sigma mr^2 = a^2.\Sigma m = \mathrm{M}.a^2$.

(3) *Moment of Inertia of a circular disc about an axis through its centre and perpendicular to the plane of the disc.*

Let the radius of the disc be a, its mass M, and its mass per unit area m. Then $\mathrm{M} = \pi a^2.m$. Consider the disc divided into a large number of thin rings concentric with the circumference of the disc. Let the radii of a typical ring be x and $x + \delta x$. The mass of this ring is $m.2\pi x.\delta x$. Its distance from the axis is x and its M.I. about the axis is $m.2\pi x.\delta x.x^2$. The M.I. of the disc $= \Sigma m.2\pi x.\delta x.x^2$

$$= 2\pi m\int_o^a x^3.dx = 2\pi m.\frac{a^4}{4} = \mathrm{M}.\frac{a^2}{2}.$$

5. Theorems of Parallel and Rectangular Axes

The Theorem of Parallel Axes.—If the M.I. of a body about an axis through its centre of mass is known the theorem of parallel axes enables us to calculate the M.I. about any parallel axis.

Let AB in Fig. 17 be the axis of rotation and I the M.I. about this axis : let ZG be a parallel axis passing through the centre

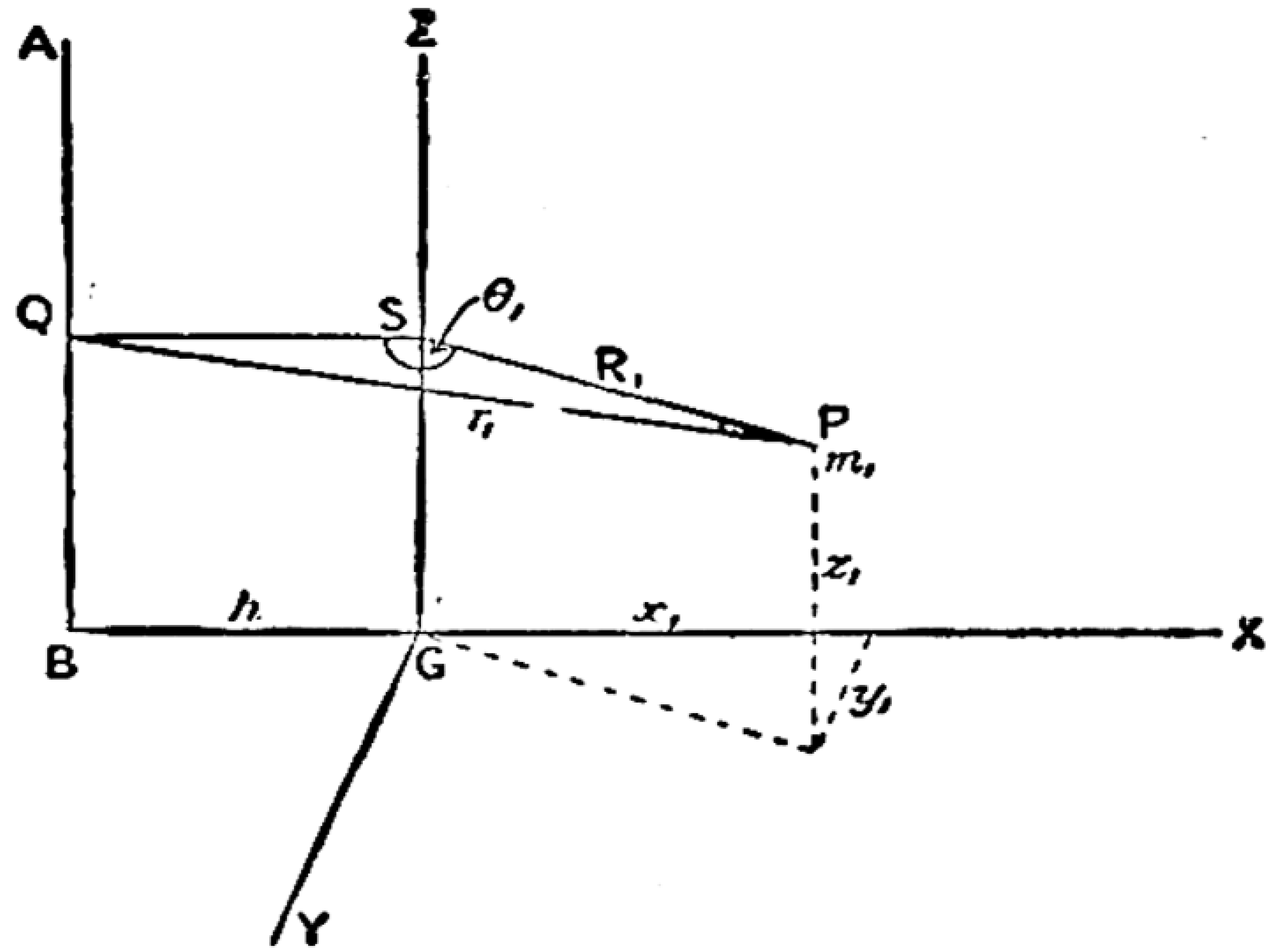

Fig. 17.

of mass G. Let the M.I. of the body about ZG be I_o. Let BG be perpendicular to ZG. Take BGX as the axis of x, GZ as axis of z, and GY perpendicular to the plane ZGX as axis of y. Let P, mass m_1 and coordinates (x_1, y_1, z_1) be the typical particle. PQ and PS, of lengths r_1 and R_1, are perpendicular to AB and ZG respectively. Let QS = BG = h.

$$\begin{aligned} \text{Then} \qquad I &= \Sigma m_1 r_1^2 = \Sigma m_1 (h^2 + R_1^2 - 2hR_1 \cos \theta_1) \\ &= Mh^2 + I_o + 2h.\Sigma m_1 x_1 \\ &= Mh^2 + I_o. \end{aligned}$$

$\Sigma m_1 x_1 = 0$, since G, the centre of mass, is at the origin.

Notice that I_o the M.I. about an axis through the centre of mass is less than the M.I. about any parallel axis.

The statement $I = I_o + M.h^2$ is the theorem of parallel axes.

The Theorem of Perpendicular Axes.—This theorem applies only to a lamina. If the M.I.s of the lamina about two axes at right angles in the plane of the lamina are known this theorem

gives us the M.I. about an axis perpendicular to the lamina through the intersection of the two axes in the lamina.

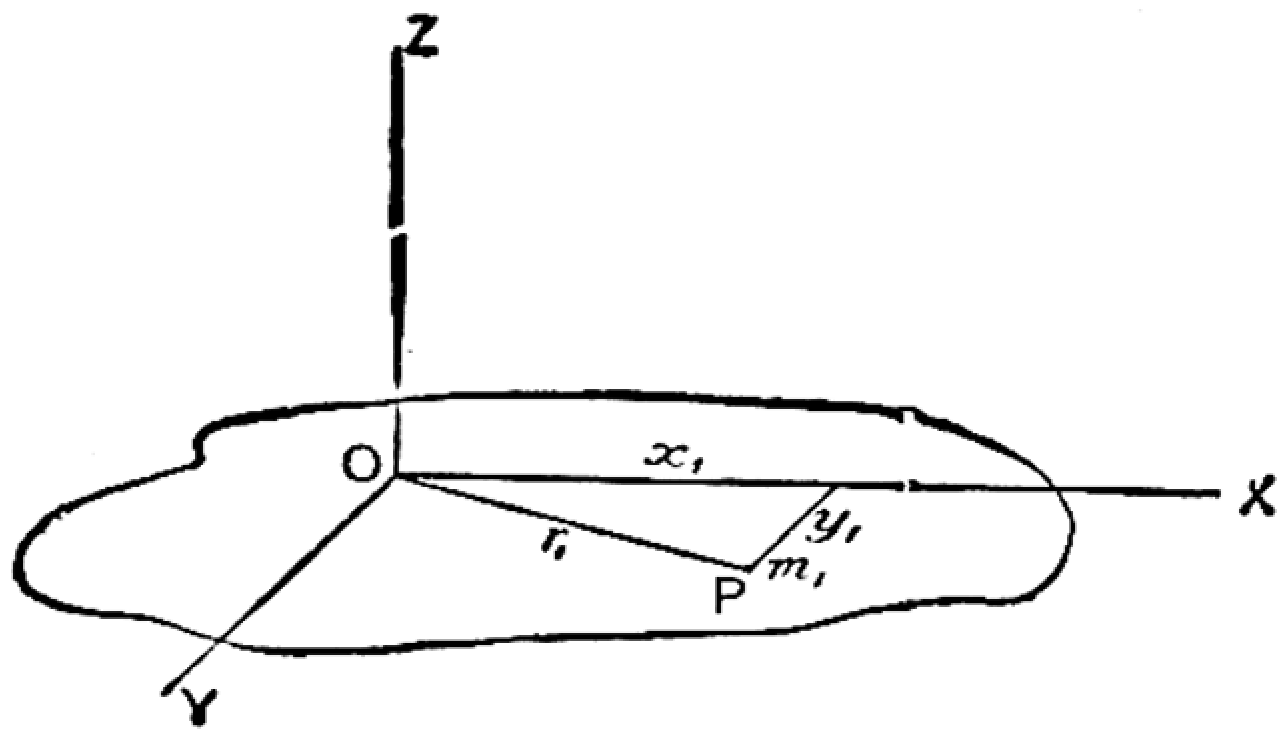

FIG. 18.

Let the M.I. about OX be I_x and about OY be I_y, XOY being a right angle. Take OX and OY as axes, and OZ perpendicular to OX and OY. Let P be the typical particle, mass m_1 and coordinates $(x_1\ y_1)$. Let OP be r_1. Then

$$\begin{aligned} I_z &= \Sigma m_1 r_1{}^2 = \Sigma m_1 x_1{}^2 + \Sigma m_1 y_1{}^2 \\ &= I_y + I_x. \end{aligned}$$

As an application of this theorem let us find the M.I. of a thin circular disc of radius a about a diameter. Let I_x and I_y denote the M.I.s about the perpendicular diameters XX^1 and YY^1. Let I_z denote the M.I. about the perpendicular axis through the centre.

From symmetry $\qquad I_x = I_y.$

From the theorem $\qquad I_x + I_y = I_z = M.\dfrac{a^2}{2}.$

Therefore $\qquad I_x = I_y = M.\dfrac{a^2}{4}.$

6. More Moments of Inertia

(1) *Rectangular Lamina and Rectangular Block.*—The M.I. of the rectangular lamina ABCD, Fig. 19, of mass M about the axis YY′ is easily seen to be $M.\dfrac{a^2}{12}$. The lamina may be imagined divided into a large number of thin strips of mass m perpendicular to YY′. The M.I. about YY′ of each of these strips will be $m.\dfrac{a^2}{12}$ and therefore the M.I. of the lamina

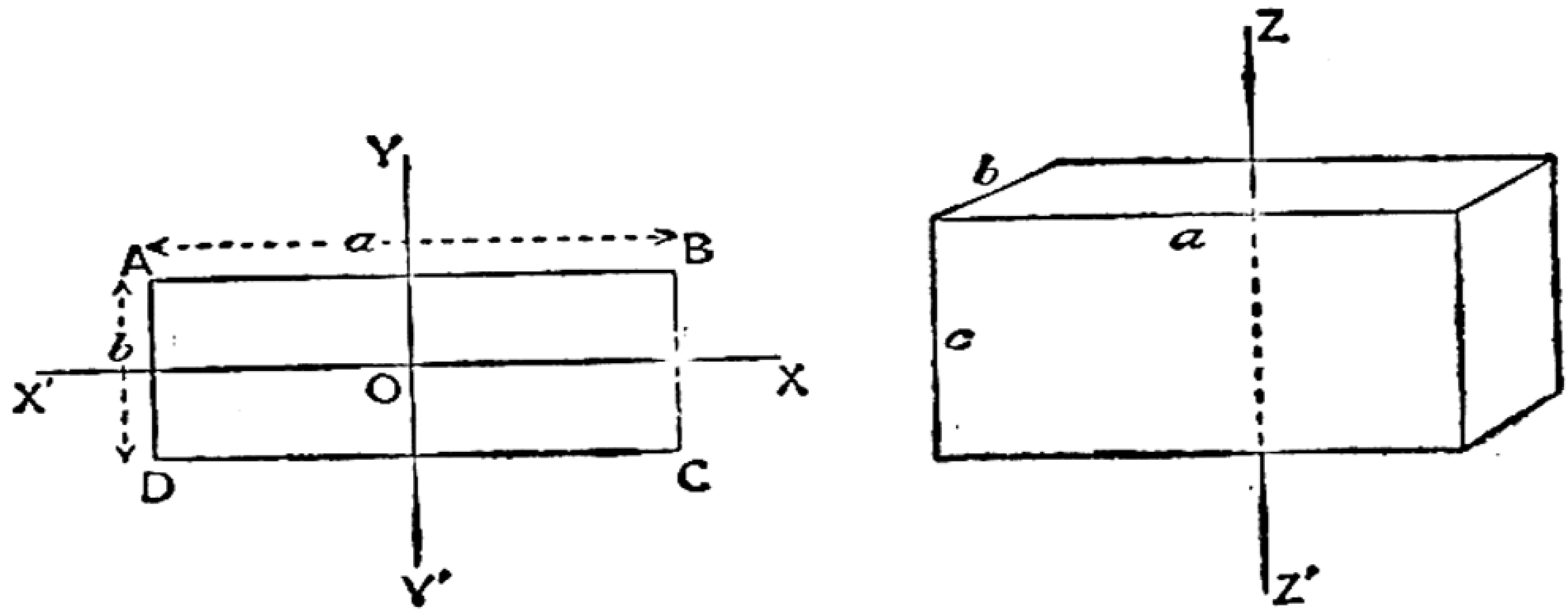

FIG. 19.

$$= \Sigma m . \frac{a^2}{12} = \frac{a^2}{12} . \Sigma m = \mathrm{M} . \frac{a^2}{12}.$$

Similarly, the M.I. about $\mathrm{XX}' = \mathrm{M} . \frac{b^2}{12}$.

The M.I. about an axis ZZ′ through O and perpendicular to XX′ and YY′ will, by the theorem of perpendicular axes, be $\mathrm{M} . \frac{a^2 + b^2}{12}$.

To find the M.I. of the rectangular block of Fig. 19 about the axis ZZ′, perpendicular to the face *ab* and passing through the centre of the block, we imagine it divided into a large number of laminæ parallel to the face *ab*. If *m* is the mass of one of these laminæ its M.I. about ZZ′ is $m . \frac{a^2 + b^2}{12}$. The M.I. of the whole block about ZZ′ is then $\Sigma m . \frac{a^2 + b^2}{12} = \mathrm{M} . \frac{a^2 + b^2}{12}$, where M is the mass of the block.

The dimension *c* of the block does not appear in the formula. It is represented in the mass M, for blocks differing only in height will have masses and therefore M.I.s proportional to their heights.

(2) *Cylinder*.—The M.I. of a cylinder of radius *a* and length *l* about its axis is by similar reasoning $\mathrm{M} . \frac{a^2}{2}$, where M is its mass. The length *l* produces its effect through the mass M.

The calculation of the M.I. of the above cylinder about the axis YY′, Fig. 20, through its centre and perpendicular to its

length affords an illustration of the use of the theorems of section 5.

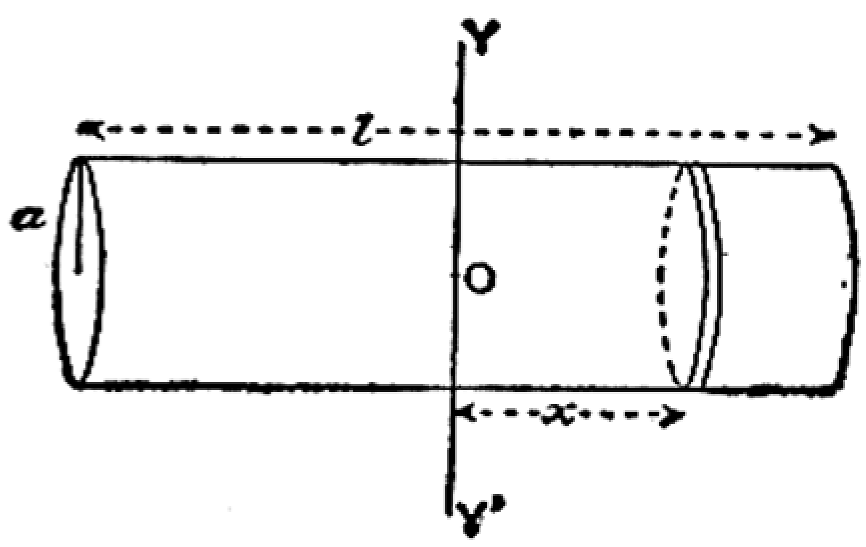

FIG. 20.

Consider a thin slice of the cylinder perpendicular to its axis whose centre is at a distance x from YY′ and whose thickness is δx. Let its mass be m. Its M.I. about an axis through its centre and parallel to YY′ is $m.\frac{a^2}{4}$. Its M.I. about YY′ is by the theorem of parallel axes $m.\frac{a^2}{4} + m.x^2 = m\left(x^2 + \frac{a^2}{4}\right)$. The M.I. of the cylinder about YY′ is $\Sigma m\left(x^2 + \frac{a^2}{4}\right)$.

If ρ is the density of the cylinder $m = \pi a^2.\delta x.\rho$. Therefore the M.I. of the cylinder

$$= \pi a^2.\rho \int_{-\frac{l}{2}}^{+\frac{l}{2}} \left(x^2 + \frac{a^2}{4}\right) dx$$

$$= \pi a^2 \rho \left(\frac{l^3}{12} + \frac{a^2.l}{4}\right) = M\left(\frac{l^2}{12} + \frac{a^2}{4}\right).$$

The M.I. of a hollow cylinder is the M.I. of the full cylinder minus that of the missing portion.

(3) *Spherical Shell and Solid Sphere about a Diameter.*— Take the origin at the centre of the spherical shell of radius a. To find the M.I. about the x axis consider the zone cut off by planes perpendicular to the x axis and distant x and $x + \delta x$ from the origin. This zone forms a thin ring of radius $\sqrt{a^2 - x^2}$. The zone area is the same as that cut off by the same planes in the circumscribing cylinder whose axis coincides with the x axis. It is therefore $2\pi a.\delta x$.

If m is the mass of the shell per unit area the M.I. of this ring about the x axis is $2\pi a.\delta x.m\,(a^2 - x^2)$,

and the M.I. of the shell $= \int_{-a}^{a} 2\pi a.m\,(a^2 - x^2)\,dx$

$$= 2\pi a.m.\frac{4a^3}{3} = 4\pi a^2.m.\frac{2a^2}{3} = M.\frac{2a^2}{3},$$

where M is the mass of the shell.

A solid sphere of radius a may be built up of a large number of thin shells. If m is the mass per unit volume, the mass of a shell of radius r and thickness δr is $4\pi r^2 . \delta r . m$. Its M.I. about a diameter is $4\pi r^2 \delta r . m . \frac{2r^2}{3}$.

Therefore the M.I. of the sphere about a diameter

$$= \Sigma\, 4\pi r^2 . \delta r . m . \frac{2r^2}{3} = \frac{8\pi}{3} . m \int_0^a r^4\, dr = \frac{8\pi}{3} . m . \frac{a^5}{5}$$

$$= \frac{4}{3}\pi a^3 . m . \frac{2a^2}{5} \qquad = \mathrm{M} . \frac{2a^2}{5},$$ where M is the mass of the sphere.

7. Measurement of Moment of Inertia by Bifilar Pendulum

The body, which we will suppose to be a hollow cylinder, is suspended by two equal, parallel threads attached at equal distances on either side of the centre of the cylinder. It is turned through a small angle θ about its centre and allowed to oscillate. The time for a number of vibrations is measured with a stop-clock and the period deduced.

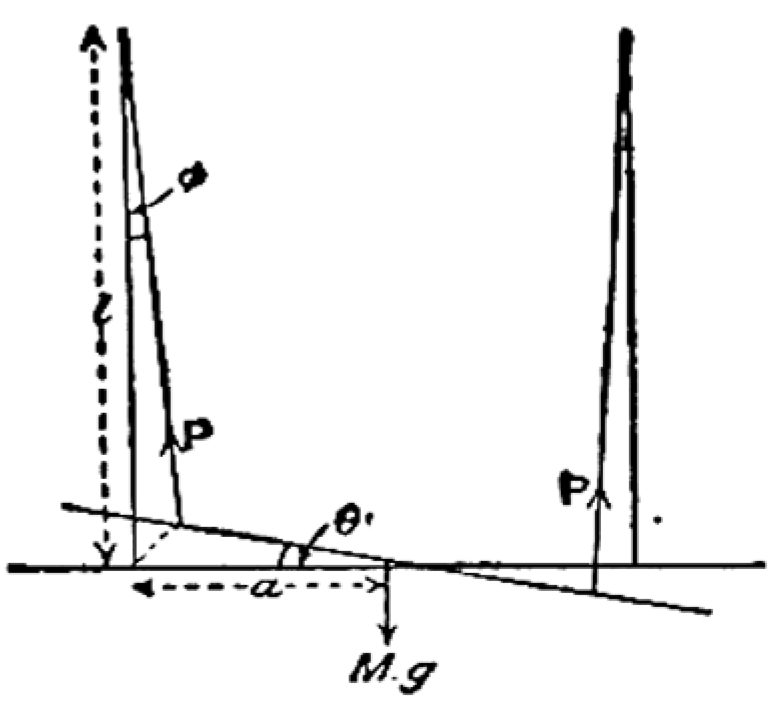

FIG. 21.

With the dimensions shown in Fig. 21 we have, since θ and ϕ are small, $a . \theta = l . \phi$. The restoring couple is P sin $\phi . 2a$

$$= \frac{1}{2} \mathrm{M}g . \phi . 2a = \mathrm{M} . g . \frac{a^2}{l} . \theta,$$

and is in the direction of θ decreasing.

$$\text{So } \mathrm{I}\ddot{\theta} = - \frac{\mathrm{M}ga^2}{l} . \theta.$$

The vibrations are thus simple harmonic, and the period $\mathrm{T} = 2\pi\sqrt{\frac{\mathrm{I} . l}{\mathrm{M}ga^2}} = 2\pi\sqrt{\frac{k^2 . l}{g . a^2}}$, where I and k are the M.I. and radius of gyration about an axis through the centre and perpendicular to the axis of the cylinder. T, l, a, M can be measured and I found from this formula.

Notice that T varies as $\sqrt{l}$, and inversely as a.

8. Action of Coplanar Forces and Impulses on a Rigid Body

A set of coplanar forces acting on a rigid body will in general produce a motion which is a mixture of translation and rotation. There will be a translational motion of the Centre of Mass of the body and a rotation of the body about its Centre of Mass. The acceleration of the Centre of Mass is that which would be obtained if the whole mass were concentrated at that point and all the forces were transferred, parallel to their original directions, to act at that point. The angular acceleration of the body about its Centre of Mass is given by the equation $I\ddot{\theta} = \Sigma F.p$, where $\Sigma F.p$ denotes the algebraic sum of the moments of the forces about an axis through the Centre of Mass and perpendicular to the plane of the forces. I is the M.I. of the body about the same axis. θ is the angle between some line in the body in the plane of the forces and some fixed direction.

An impulsive force is one of the nature of a blow or jerk. The force acts for a short time only and is variable. Both the time of action and the strength of the force are usually unknown. The effect produced is a sudden change in momentum of the body on which the impulsive force acts. If the force is steady $F \times t$, or if variable $\int_0^t F.dt$, where t is the time the force acts, is called the Impulse and, in the case of a massive particle, is equal to the change in momentum produced.

The action of impulsive forces on a rigid body is similar to that of steady forces. The resulting velocity of the mass-centre is the same as if the whole mass were concentrated at that point and the impulsive forces acted on it parallel to their original directions. The angular momentum of the body is given by $I.\omega = \Sigma P.p$, where $\Sigma P.p$ denotes the algebraic sum of the moments of the impulses about an axis through the mass-centre and perpendicular to the plane of the impulses.

To illustrate this Fig. 22 (a) represents a rod, length a and mass M, resting on a smooth table. Its mass-centre G will begin to move in the direction of the force F with acceleration F/M. At the same time the rod will rotate about G in an anti-clockwise direction with angular acceleration $\frac{F.p}{I}$ or $\frac{12F.p}{M.a^2}$.

That this is the effect of the force F becomes clear if two equal and opposite forces identical with F are introduced at G. These two forces have no effect on the motion, but we now have a force F at G, and a couple of moment $F.p$.

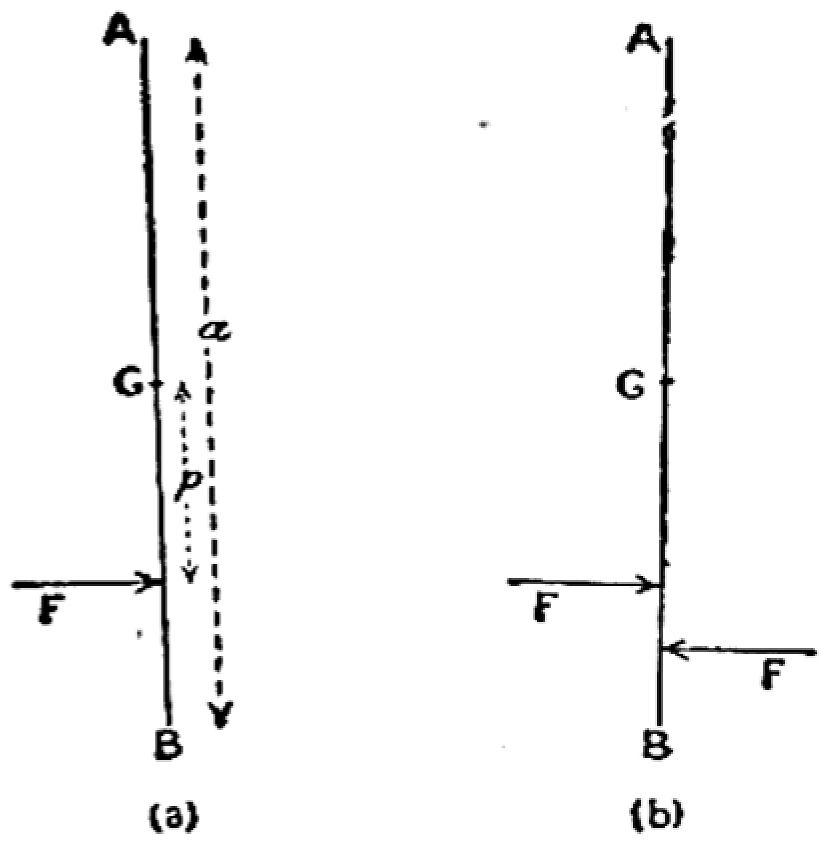

FIG. 22.

If the force F in Fig. 22 (a) is replaced by an impulse P, G will move in the direction of the impulse with initial velocity P/M and the rod will start to rotate about G with angular velocity $\frac{P.p}{I}$.

Under the action of the forces in Fig. 22 (b) the mass-centre G will remain at rest and the body will rotate about that point.

9. Conservation of Angular Momentum

If a rigid body, or a system of particles, in motion be free from external forces then its angular momentum about any axis is constant. Also, if external forces exist but are such that the sum of their moments about an axis is zero, then the angular momentum about that axis is constant. This follows from the equation of the last section, $I.\ddot{\theta} = \Sigma F.p$. If $\Sigma F.p = 0$, then $I\ddot{\theta} = 0$ and $I\dot{\theta}$ is therefore constant.

This principle is of great importance and often provides a short cut to the solution of problems on the motion of rigid bodies.

10. Conservation of Energy

A moving body in general has mechanical energy in the two forms of Potential Energy (P.E.) and Kinetic Energy (K.E.). It may have P.E. on account of the earth's gravitational field or on account of a state of strain in the body itself. In the first case its P.E. in absolute units is $M.g.h$, where M is the mass of the body, h is the height of its mass-centre above some standard level, and g is the acceleration due to gravity.

Its K.E. is the sum of the kinetic energies of its individual particles. This, for a rigid body, is equal to the K.E. of a mass M moving with the mass-centre and the K.E. of rotation of the body about its mass-centre.

The whole K.E. $= \frac{1}{2} M.v^2 + \frac{1}{2} I.\omega^2$, M being the mass of the body, v the velocity of its mass-centre, I its M.I. about the axis of rotation through its mass-centre, and ω the angular velocity about this axis.

The Principle of the Conservation of Energy has been stated in various forms :—

The total amount of energy in the universe is constant.

Energy cannot be created or destroyed, but is only transformed from one kind into another at certain fixed rates of exchange.

For instance, $4{\cdot}2\ 10^7$ ergs $= 1$ calorie,
$4{\cdot}2$ joules $= 1$ calorie.

The Principle is a sweeping generalisation from a few simple experiments. Perhaps the chief evidence in its favour is that so far no exceptions to it have been encountered. It has, however, always been difficult to account for the enormous output of energy by stars like the sun. This is now explained by a continual diminution of the star's mass, matter being a form of energy. Einstein's Relativity Theory indicates an equivalence of 1 gram to C^2 ergs, $C = 3 \times 10^{10}$ being the velocity of light *in vacuo* in cm. per sec.

In Mechanics the application of the Principle is restricted to the transformations between Potential Energy and Kinetic Energy.

If work is done on a system of bodies by external forces there is a gain in energy of the system equal to the work done. If the system does work on bodies external to itself it loses an equal amount of energy. If the system is isolated its energy remains constant.

When work is done against frictional forces between bodies in the system, there is, however, a loss of mechanical energy, an amount being transformed into Heat, Sound, etc. This is also the case when imperfectly elastic bodies of the system impinge on one another. The energy transformed into Heat and Sound in the bodies becomes vibrational energy of the atoms or molecules of which the bodies are composed, and is mechanical energy. But this vibration of the particles of bodies is incompatible with our definition of a rigid body, and is not considered in the less general form of the Principle in Mechanics.

The mechanical form of the Principle is applied then only in cases where the amount of energy transformed into Heat, Sound, etc., is small or negligible. In fact, we assume that no such transformation occurs, and we cannot therefore apply the Principle to problems in which friction and imperfectly elastic impacts are known to play a considerable part.

The most common application of the Principle is to systems under the influence of gravity. Here the work done by the force

of gravity is equal to the energy gained by the system. This equality is often called the energy equation. If we consider the earth as part of the system, then the sum of the P.E. and K.E. is constant, and any decrease in P.E. is accompanied by an equal increase in K.E.

Example.—Disc Rolling Down Inclined Plane.—The disc starts from rest. Let the velocity of its centre after falling through a height h be v. It is assumed that there is no slipping. Then the angular velocity of the disc about its centre will be $\omega = \frac{v}{a}$, when the velocity is v, a being the radius.

$$\text{The loss of P.E.} = M.g.h.$$

$$\begin{aligned}\text{The gain of K.E.} &= \frac{1}{2}M.v^2 + \frac{1}{2}I.\omega^2\\ &= \frac{1}{2}M.v^2 + \frac{1}{2}M.\frac{a^2}{2}.\frac{v^2}{a^2}\\ &= \frac{3}{4}M.v^2.\end{aligned}$$

So $$v^2 = \frac{4}{3}g.h.$$

If f is the acceleration down the incline

$$v^2 = 2.f.\frac{h}{\sin\alpha} \text{ (Fig. 23)}$$

Whence, $$f = \frac{2}{3}g\sin\alpha.$$

For a sphere $$I = \frac{2}{5}M.a^2,$$

and we have $$M.g.h = \frac{1}{2}Mv^2 + \frac{1}{2}M.\frac{2}{5}a^2.\frac{v^2}{a^2}$$

$$i.e., \frac{7}{10}v^2 = g.h \text{ or } v^2 = \frac{10}{7}g.h,$$

and the acceleration down the slope $= \frac{5}{7}g\sin\alpha$. (The acceleration of a particle down a smooth inclined plane of angle α is $g.\sin\alpha$.)

The acceleration of the sphere down the plane may also be found by considering the actual forces on the sphere. They are shown in Fig. 23. F is a friction force, but since there is no sliding no work is done against friction.

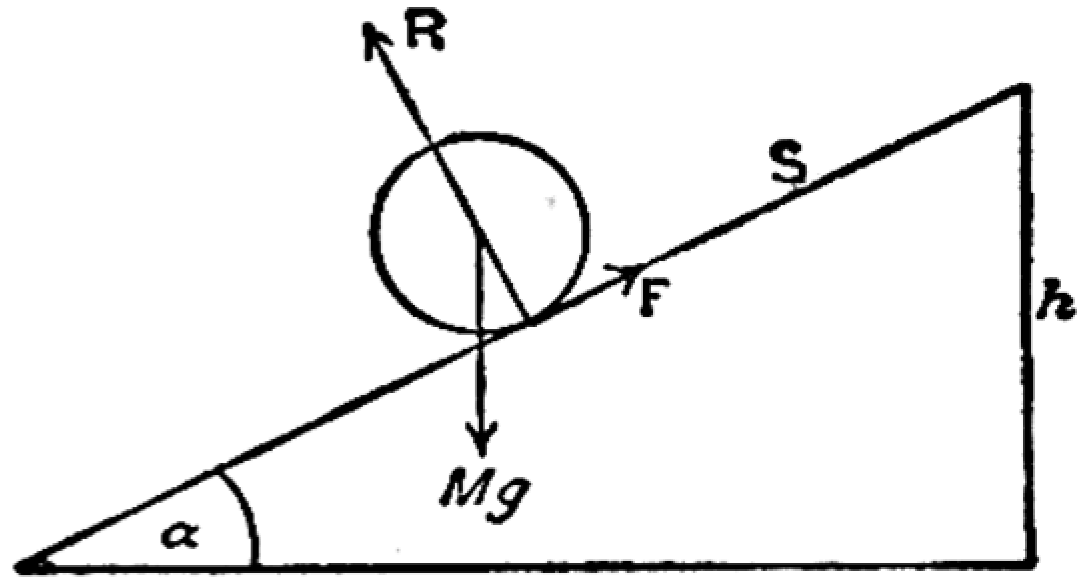

FIG. 23.

We have then $s = a.\theta$ and $\ddot{\theta} = \dfrac{\ddot{s}}{a}$, where θ is the angle the sphere has turned through in describing the distance s.

For the motion down the plane $M.\ddot{s} = Mg \sin\alpha - F$.

For the rotation $\qquad I.\ddot{\theta} = F.a$.

$$i.e.,\ M\ddot{s} = Mg\sin\alpha - \frac{I\ddot{\theta}}{a} = Mg\sin\alpha - \frac{2}{5}Ma^2.\ddot{s}.\frac{1}{a^2}$$

$$\therefore \frac{7}{5}\ddot{s} = g\sin\alpha$$

or
$$\ddot{s} = \frac{5}{7}g\sin\alpha.$$

11. The Compound Pendulum

In calculating the period $T = 2\pi\sqrt{\dfrac{l}{g}}$ for the simple pendulum we assume the string to be weightless and the bob to be a massive particle. In the compound pendulum the string is replaced by a rigid rod whose mass is not negligible, and the massive particle by a disc or sphere of finite size. Before we can calculate the period of the compound pendulum of the above particular shape it is necessary to consider the case of a plane lamina oscillating in a vertical plane about a horizontal axis under the influence of gravity. The lamina (Fig. 24) oscillates about an axis through O perpendicular to its plane. Let G be the centre of gravity of the lamina and $OG = h$. At some instant let θ be the inclination of OG to the vertical. The

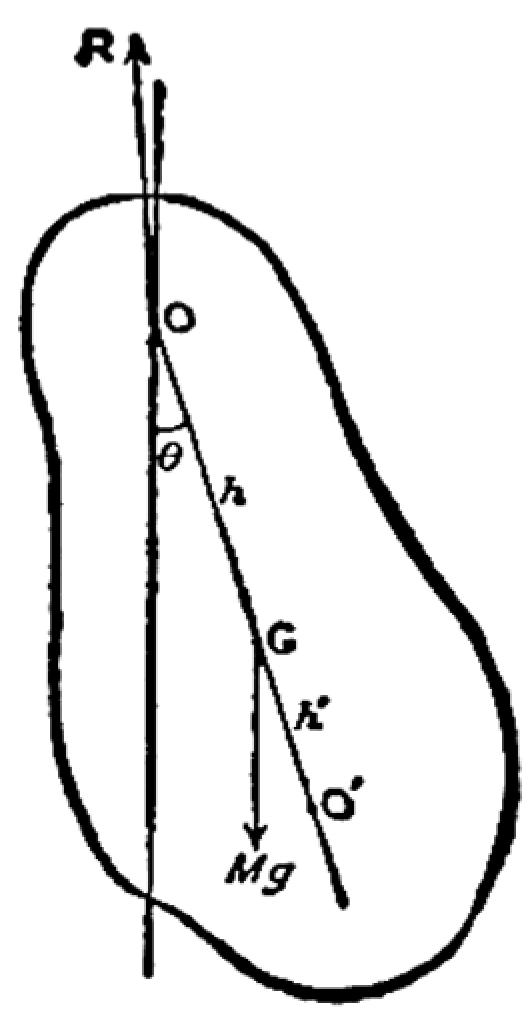

FIG. 24.

D

forces acting on the lamina are its weight Mg and the reaction R from the axis. For the present the precise direction of R is unimportant, as it has no moment about the axis. The angular acceleration will be given by $I\ddot{\theta} = -Mg.h.\sin\theta$, where I is the M.I. of the lamina about O. The negative sign is present because θ is increasing as the lamina rotates in an anti-clockwise direction, and $I\ddot{\theta}$ is equal to the sum of the moments about the axis in the direction of θ increasing.

If the amplitude of the oscillation is small we may write $I\ddot{\theta} = -Mg.h.\theta$.

The motion is therefore simple harmonic, and

$$T = 2\pi\sqrt{\frac{I}{M.g.h}} = 2\pi\sqrt{\frac{k^2}{h.g}},$$

where k is the radius of gyration of the lamina about O.

This expression is of the same form as that for the period of the simple pendulum. If $\frac{k^2}{h} = L$, then $T = 2\pi\sqrt{\frac{L}{g}}$. L is the length of the equivalent simple pendulum, *i.e.*, one having the same time of oscillation.

The point O′, distant $L = \frac{k^2}{h}$ from O along OG, is called the Centre of Oscillation of the lamina oscillating about O.

12. The Centres of Oscillation and Suspension are Interchangeable

Let K be the radius of gyration of the lamina about an axis perpendicular to its plane through its centre of gravity G, then $k^2 = K^2 + h^2$, since

$$I_O = I_G + M.h^2 \text{ or } M.k^2 = M.K^2 + M.h^2.$$

Also let k' be the radius of gyration about an axis perpendicular to the lamina through O′, and $GO' = h' = L - h$.

Then $k'^2 = K^2 + h'^2 = K^2 + (L-h)^2 = K^2 + L^2 - 2Lh + h^2$.

Let the equivalent simple pendulum for small oscillations about O′ be L′.

Then $L' = \frac{k'^2}{h'} = \frac{k^2 + L^2 - 2Lh}{L-h} = \frac{k^2}{L-h} + L - \frac{Lh}{L-h} = L,$

since $k^2 = L.h$.

Thus the times of oscillation about O and O′ are equal.

Notice that $$L = \frac{k^2}{h} = \frac{K^2 + h^2}{h} = \frac{K^2}{h} + h,$$

$$i.e.,\ h' = \mathrm{L} - h = \frac{\mathrm{K}^2}{h}$$

or

$$\mathrm{K}^2 = h.h'.$$

The interchangeability of O and O′ follows also from the symmetry of this relation, since

$$\mathrm{L} = \frac{k^2}{h} = \frac{\mathrm{K}^2 + h^2}{h} = \frac{hh' + h^2}{h} = h' + h,$$

and

$$\mathrm{L}' = \frac{k'^2}{h'} = \frac{\mathrm{K}^2 + h'^2}{h'} = \frac{hh' + h'^2}{h'} = h + h'.$$

Example 1.—A thin rod is suspended at various points in its length and allowed to oscillate through a small angle under the influence of gravity. How does the period vary with the position of the point of suspension ?

Let the rod be AB of length l, and let the point of suspension be O, distant h from G the middle point of the rod. Then, with our notation,

$$\mathrm{T} = 2\pi\sqrt{\frac{k^2}{h.g}},$$

and

$$\frac{g\mathrm{T}^2}{4\pi^2} = \frac{k^2}{h} = \frac{\mathrm{K}^2}{h} + h.$$

If T^2 is plotted against h the resulting curve is a hyperbola in shape, but since T^2 cannot be negative the graph will be as shown in Fig. 25. The time of oscillation about G is infinite.

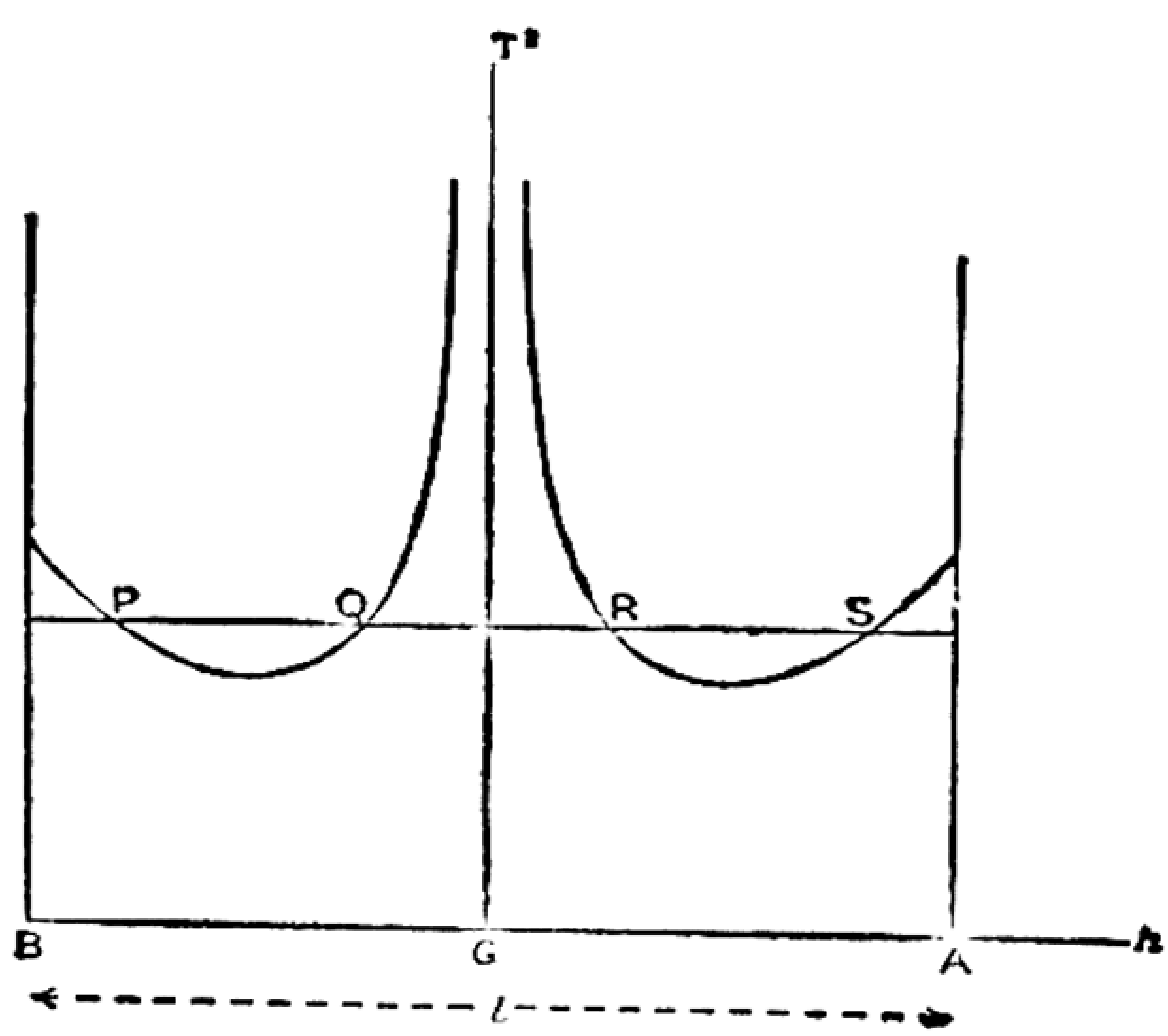

FIG. 25.

A straight line parallel to BA may cut the curve in four points such as P, Q, R, S, so that for a certain range of values of T there can be found in the rod four points of suspension giving the same period. The length PR, which is equal to QS, is the length of the equivalent simple pendulum for the point of suspension P.

Also, since $$\frac{gT^2}{4\pi^2} = \frac{K^2}{h} + h,$$

T^2 will be a minimum when $-\frac{K^2}{h^2} + 1 = 0$, *i.e.*, when $h = K$.

The equivalent simple pendulum in this case is

$$2K = \frac{l}{\sqrt{3}}.$$

When the rod is suspended at its end the equivalent simple pendulum is $\frac{k^2}{h} = \frac{l^2}{3} \times \frac{2}{l} = \frac{2}{3}.l.$

Example 2.—A sphere 10 cm. diameter is suspended from a fine wire 95 cm. long, of negligible mass. Find the length of the equivalent simple pendulum.

Here $k^2 = K^2 + h^2 = \frac{2}{5}.5^2 + 100^2$ since $h = 100$ cm.

So $$L = \frac{k^2}{h} = \frac{10{,}010}{100} = 100{\cdot}1 \text{ cm.}$$

With a bob 2 cm. diameter and a wire 99 cm. long,

$$L = 100{\cdot}004 \text{ cm.}$$

For a bob 10 cm. diameter and a wire 45 cm. long,

$$L = 50{\cdot}2 \text{ cm.}$$

In the measurement of g by means of the simple pendulum it is clear that the error in taking the length of the pendulum to the centre of the sphere instead of to its centre of oscillation is negligible if the sphere is less than 2 cm. diameter.

13. Diagram of Compound Pendulum Properties

In Fig. 26 O is the point of suspension and G the centre of gravity of the lamina, OG $= h$.

Draw GP $=$ K at right angles to OG.

Join OP, and draw PO′ perpendicular to OP, to meet OG produced in O′.

Then O′ is the centre of oscillation.

To prove this we note that $OP^2 = K^2 + h^2$.

Therefore $OP = k$, the radius of gyration about O. Also from the geometry of the right-angled triangles,

$$OO'.OG = OP^2,$$

i.e., $$OO' = \frac{k^2}{h} = L.$$

O′ is therefore the centre of oscillation.

It is clear also that $O'P = k'$, the radius of gyration about O′.

The diagram gives a useful idea of the relative sizes of the various lengths involved and of their changes as the point of suspension O is altered.

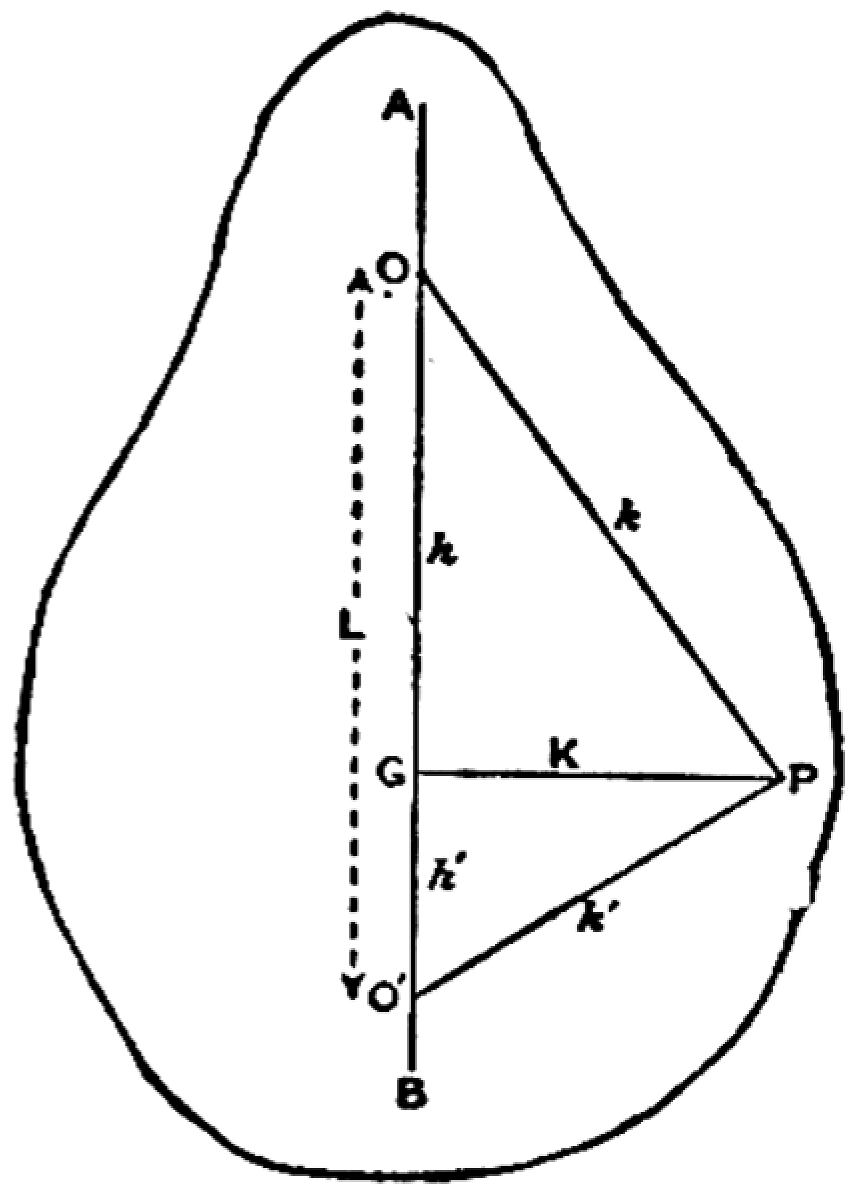

FIG. 26.

The various relations between the quantities are obvious from the geometrical properties of the three right-angled triangles.

14. Compound Pendulum with Inclined Axis of Rotation

Let COD be the axis, inclined at angle α to the horizontal, about which the oscillation occurs. AB represents the pendulum. The motion takes place in a plane through O perpendicular to CD. Let θ be the small angular displacement in this plane of the pendulum from its equilibrium position at time t. The weight Mg may be resolved into components $Mg \sin \alpha$ parallel to the axis CD and $Mg \cos \alpha$ in the plane of oscillation. The component $Mg \sin \alpha$ has no moment about CD.

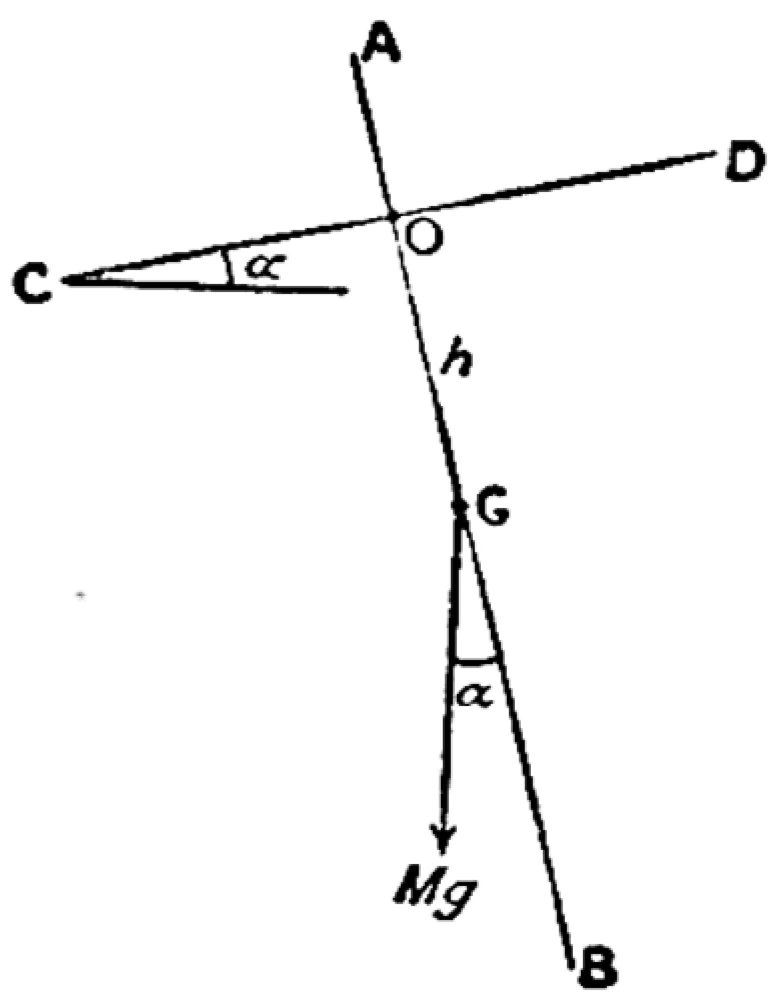

FIG. 27.

So $$I\ddot{\theta} = -Mg \cos \alpha . h \sin \theta,$$
$$= -Mg \cos \alpha . h . \theta,$$

and $$T = 2\pi\sqrt{\frac{I}{Mg \cos \alpha . h}} = 2\pi\sqrt{\frac{k^2}{h . g . \cos \alpha}}.$$

Thus $T^2 \propto \sec \alpha$.
When $\alpha = 90°$, T is infinite.

Self-closing gates are sometimes suspended from hinges which are a few degrees out of the vertical. The motion of such a gate is similar to that of a compound pendulum with inclined axis of rotation.

15. Centre of Percussion

Consider a thin rod, at rest and free to move, to receive at O′ an impulse P at right angles to its length. The line AB of Fig. 26 can represent the rod. Let M be its mass, G its mass-centre, and $O'G = h'$.

The mass-centre G will be given a velocity u in the direction of P such that $P = M.u$. The rod will also receive an angular velocity ω about G such that $M.K^2\omega = P.h'$.

A point O situated on the other side of G at a distance h from G will have a forward velocity u (in the direction of P) and a backward velocity $h.\omega$ due to the rotation of the rod about G. Its resulting initial velocity will be

$$u - h.\omega = \frac{P}{M} - h.\frac{Ph'}{MK^2}$$
$$= \frac{P}{M}\left(1 - \frac{h.h'}{K^2}\right)$$

This will be zero if $h.h' = K^2$, *i.e.*, provided the points O and O′ are corresponding centres of suspension and oscillation for the rod as a compound pendulum.

If the rod is supported on an axis through O the impulsive action between this axis and its supports will be zero if O is such that $h.h' = K^2$. O′ is called the Centre of Percussion under these conditions. No jar is experienced by the support if the impulse is delivered at the corresponding centre of percussion. In batting this is the right spot on which to receive the ball.

It is interesting, too, that the cushions of a billiard table are placed so as to be on the level of the centre of oscillation of the billiard ball for a tangential axis.

Examples

1. Find an expression for the moment of inertia of a door of mass M lb. of height h ft. and breadth b ft. about the axis through the hinges.

2. Find an expression for the moment of inertia of a thin uniform rod about an axis perpendicular to it and intersecting it at one-quarter of its length from one end.

If the diameter of a penny is 3 cm. and its mass is 9 gm., calculate its total energy when it rolls along a table with velocity 6 cm. per sec.

3. Explain what is meant by 'moment of inertia' and 'angular momentum.'

A thin hollow cylinder open at both ends and weighing 96 lb. (*a*) slides with a speed of 10 *f.p.s* without rotating, (*b*) rolls with the same speed without slipping. Compare the kinetic energies of the cylinder in the two cases. (N.U.)

4. A block of wood slides down an inclined plane inclined at 30° to the horizontal with the same acceleration as a cylinder rolls down a second inclined plane. If the coefficient of friction between the wood and the plane is ·3, calculate the angle of inclination of the second plane. (C. Schol.)

5. What do you understand by 'moment of inertia'? How do you account for the fact that a long uniform rod suspended from one end oscillates more rapidly than a simple pendulum of the same length? Calculate the length of the equivalent simple pendulum in terms of the length of the rod. (N.U.)

6. A thin rod, 1 metre long, oscillates under the influence of gravity about a fixed axis through one end. What will be its period? (N.U.)

7. If you were provided with a large circular metal sheet of uniform thickness and with arrangements for suspending it so that it could oscillate freely, under the influence of gravity, in its own plane about various horizontal axes perpendicular to that plane, how would you expect the time of oscillation to vary with the distance of the axis from the centre of the circular sheet?

Draw a rough graph of the relation and offer some theoretical explanation. (N.U.)

8. Describe with the help of a rough graph how the period of oscillation of a uniform bar pendulum will vary as the point of suspension is moved from one end of the bar to the other.

Prove that the period of a thin circular hoop about an axis through the circumference and perpendicular to the plane of the hoop is the same as that of a simple pendulum of length equal to the diameter of the hoop. (N.U.)

9. A uniform bar of length $2l$ oscillates about a horizontal

axis distant a from its centre. Show that the periodic time of its oscillation is $2\pi\sqrt{\frac{a}{g} + \frac{l^2}{3a.g}}$. (C. Schol.)

10. A uniform thin rod of length $2l$ executes small oscillations about a horizontal axis at a distance h from the centre of the rod. Find the value of h for which the periodic time is a minimum. (C. Schol.)

11. A smooth inverted cone of height h is held with its axis vertical and its apex resting on a horizontal plane. A heavy particle is projected tangentially into the cone with a horizontal velocity V at a distance d from the plane. Show that the angular momentum of the particle about the axis is constant.

Show that the particle will just reach the top of the cone if $V = h.\sqrt{\frac{2g}{h+d}}$.

CHAPTER IV

GRAVITATION. 'g' AND 'G'

1. Direct Methods of Measuring the Acceleration of Gravity

THE earth exerts a force of attraction on all bodies. This force acting on a body is directed towards the earth's centre, and is called the Weight of the body. The strength of this force is different for different bodies, but for a particular body is constant so long as the displacements of the body are small compared with the earth's dimensions.

If the body is free to move this force will give it an acceleration towards the earth's centre. If the gravitational force on a particular body of mass M be W, then the acceleration of this body will be W/M.

It was shown experimentally by Galileo, at Pisa, that bodies of different masses fall with nearly the same acceleration, the differences being accounted for by the resistance of the air. Newton, in his Guinea and Feather experiment, allowed the bodies to fall in a vacuum, and demonstrated the equality of their accelerations. Newton also confirmed the discovery of Galileo and Huyghens that the time of swing of a simple pendulum of given length is independent of the mass of the bob, and pointed out that this fact is accurate evidence of the proportionality of Mass and Weight. For a bob of mass M and weight W the period $T = 2\pi\sqrt{\frac{M . l}{W}}$, where l is the fixed length of the pendulum. (This may easily be shown by the method of Chapter II, section 4.) Since T is independent of M, the ratio M/W must be the same for all bodies in the same locality. If the force or weight W is expressed in absolute units, $\frac{W}{M}$ is the acceleration of all bodies falling freely in this locality, and is called the acceleration due to gravity. It is denoted by g.

Some account of the variation of g over the earth's surface is given in later sections. Accurate measurements of its value have been made in different parts of the world.

A knowledge of g in all localities is essential for the accurate comparison of forces measured at different places in terms of weight. The variation of g is of importance also in determining the shape of the earth.

All accurate measurements of g have been made indirectly by pendulum methods. The direct methods to be described next may give results differing from the true value by more than 1%. In recent years electrical methods of timing have been greatly improved and direct methods for measuring g could be devised giving greater accuracy than 1%, but they are not yet school laboratory experiments.

(1) *The Falling Plate Method.*—A rectangle of plate-glass about 10×3 in. is lightly smoked on one side by holding it over a turpentine flame. It is supported by a cotton thread in wooden guides, in the manner of the guillotine, with the long sides vertical, and the bottom about 8 in. above rubber stops on the base of the frame. A tuning fork of known frequency, say 256, with a short fine bristle attached to one prong, is supported horizontally in a stand with the end of the bristle just touching the smoked surface of the glass. The fork is supported so that its vibrations are in a horizontal plane. The tuning fork is set into vibration and the cotton is burnt. The plate falls freely and a wave trace is made on the glass.

The beginning of the trace will usually be marked by a thick line and the first few waves will be difficult to distinguish. Take three points A, B, C (intersections of the wave trace with its central line) so that A is near the beginning of the trace and C about two-thirds along it, and such that AB and BC contain the same number n of half-waves. Let τ be the half-period of the fork. Then AB and BC are described in the time $n.\tau$. Let the distances of A, B, C from the starting point be s_1, s_2, s_3. These cannot be measured accurately, but their differences can.

Let the times from the start to A, B, C be t_1, t_2, t_3.

Then $$s_1 = \frac{1}{2}g.t_1^2,\ s_2 = \frac{1}{2}g.t_2^2,\ s_3 = \frac{1}{2}g.t_3^2.$$

Therefore $$\text{AB} = s_2 - s_1 = \frac{g}{2}(t_2^2 - t_1^2) = \frac{g}{2}(t_2 - t_1)(t_2 + t_1)$$

$$= \frac{g}{2}.n.\tau(t_2 + t_1).$$

Similarly, $$\text{BC} = \frac{g}{2}.n.\tau(t_3 + t_2).$$

Therefore $BC - AB = \frac{g}{2}.n.\tau(t_3 - t_1) = \frac{g}{2}.n.\tau.2\,n.\tau,$

i.e., $\qquad BC - AB = g.n^2.\tau^2.$

AB and BC can be measured with a travelling microscope. Different sets A, B, C may be chosen on the wave trace and a mean value taken for g.

(2) *Atwood's Machine.*—Consider two masses, m_1 and m_2 ($m_1 > m_2$), suspended over a light, frictionless pulley by a light, inextensible string. Let T be the tension of the string whilst the system is in motion. If f is the acceleration of m_1 downwards we have $m_1g - T = m_1.f$, and since f is also the acceleration of m_2 upwards, $\qquad T - m_2g = m_2.f.$

Whence, by addition $$f = \frac{m_1 - m_2}{m_1 + m_2}.g \quad . \quad . \quad . \quad . \quad (1)$$

and $$T = m_2(f + g) = \frac{2m_1.m_2}{m_1 + m_2}.g \quad . \quad . \quad . \quad . \quad (2)$$

Equation (1) gives g if the small acceleration f can be measured directly.

In Atwood's machine two equal masses P, P, in the form of cylinders are suspended over a light pulley. In some forms the axle of the pulley rests on four friction wheels, a device which reduces the retarding moment due to friction. The vertical pillar supporting the pulley is graduated in centimetres and feet and carries a fixed platform A, a movable ring B, and a second platform C, also movable. The platform A can be dropped very quickly. A mass Q is added to the upper mass P just sufficient to balance the friction of the system. This mass Q has to be found by trial, so that the system, when set in motion, continues to move with uniform velocity. A second mass R, the rider, of special shape shown in Fig. 28, is placed on Q. The platform A is dropped and the system moves with constant acceleration due to the rider R, until the ring is reached. Here R is removed by the ring and the system moves from B to C with constant velocity. The time from B to C is measured by stop-watch. The distances AB and BC are also measured. To AB should be added the length of P + Q. Through this total distance d the system moves with acceleration f. The velocity acquired is given by $v^2 = 2f.d$.

If the distance BC (or rather BC less the length of P + Q) is h, and is described in time t secs., then

$$v = \frac{h}{t} \text{ and } f = \frac{v^2}{2d} = \frac{h^2}{2d.t^2}.$$

But
$$f = \frac{\mathrm{R}}{2\mathrm{P} + \mathrm{Q} + \mathrm{R}}.g.$$

So
$$g = \frac{2\mathrm{P} + \mathrm{Q} + \mathrm{R}}{\mathrm{R}}.\frac{h^2}{2d.t^2},$$

in terms of the known masses and measured distances and time.

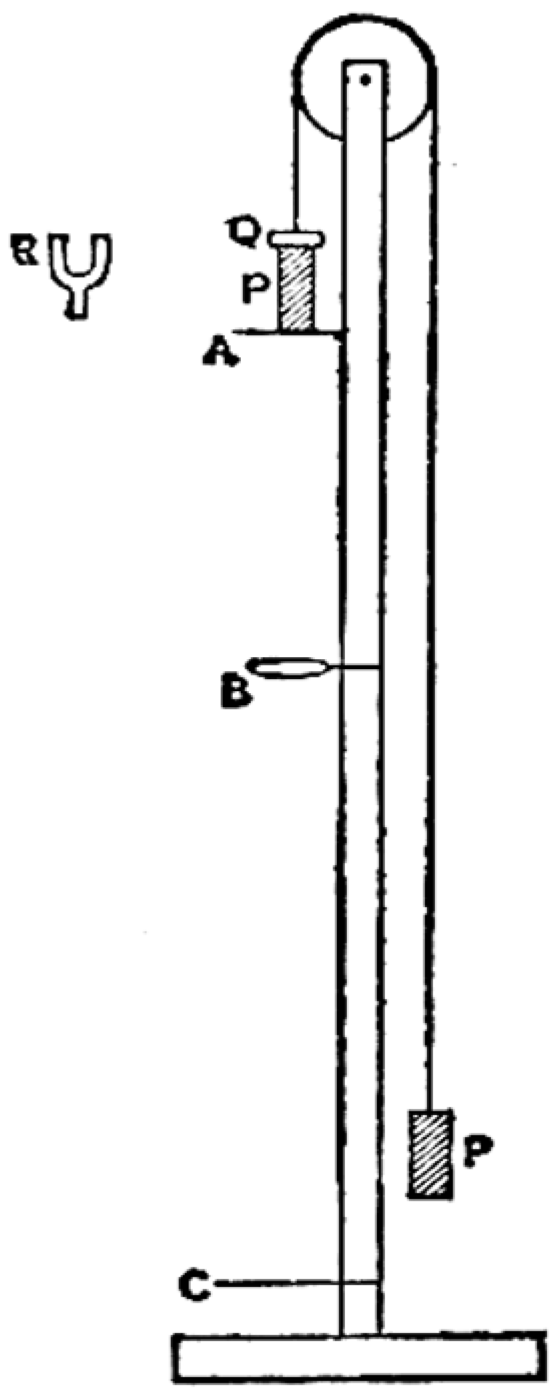

FIG. 28.

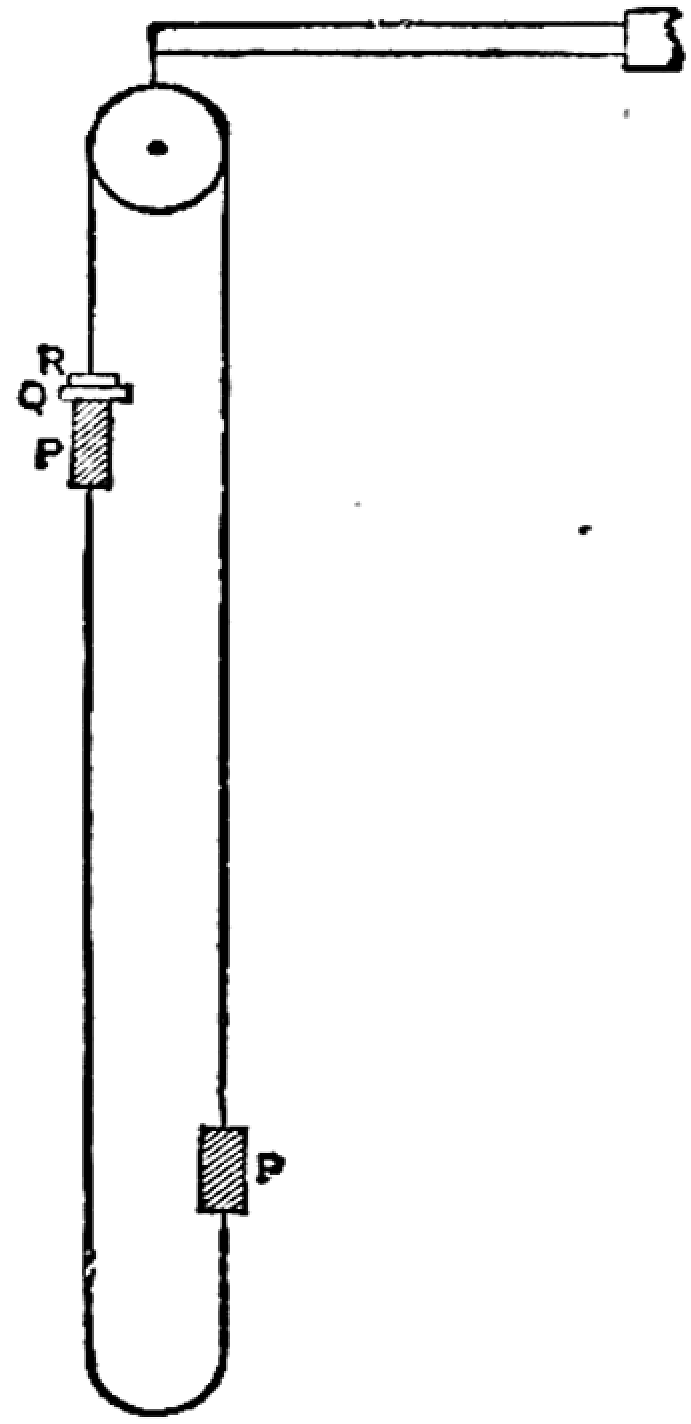

FIG. 29.

There are several sources of error in this method of finding g. No allowance has been made for the inertia of the pulley. (This can be done by adding to $2\mathrm{P} + \mathrm{Q} + \mathrm{R}$ the quantity I/a^2, a being the radius of the wheel and I its M.I.) Timing by stop-watch for a period of 10 or 20 seconds may be in error by as much as 4%. The weight of the string is neglected.

A later form of the machine, Fig. 29, has a continuous ribbon of paper, about an inch wide, to which the weights are attached. The timing is carried out by means of a vibrating strip of steel, clamped at one end and carrying at the other a fine paint-brush supplied with ink. The brush end of the strip is drawn a little

to one side. The pulley and strip are released at the same moment, and as the system moves a wave trace is drawn on the paper ribbon. From this the acceleration may be found as in the Falling Plate Experiment. The beginning of the trace is clearer in this Atwood machine experiment owing to the longer period of the vibrator, $\frac{1}{5}$ or $\frac{1}{10}$ sec.

2. Pendulum Methods. Borda's Pendulum

The formula for the period of a pendulum, $T = 2\pi\sqrt{\frac{k^2}{h.g}}$, contains g and quantities which are determined by the dimensions of the pendulum. T can be measured with great accuracy.

One of the early measurements leading to a determination of g was made by Borda, in Paris, in 1792, using a pendulum about 2 metres long of fine wire, carrying a sphere of platinum about 10 cm. in diameter. The other end of the wire was fastened to a knife-edge having as nearly as possible the same period as the pendulum. The period of the pendulum was approximately 4 secs. In calculating k and h the mass of the wire and the knife-edge can be neglected. If l is the distance from the knife-edge to the centre of the sphere and r the radius of the sphere, $k^2 = l^2 + 2r^2/5$ and $h = l$,

so
$$T = 2\pi\sqrt{\frac{l^2 + \frac{2r^2}{5}}{l.g}}.$$

The observed period requires correction on account of (a) the finite angle of swing, (b) the buoyancy of the air, (c) the small mass of air which is carried along with the pendulum as it swings.

(a) If T_0 is the period corresponding to an infinitesimal amplitude the observed period $T = T_0\left(1 + \frac{\alpha^2}{16}\right)$ or $T = T_0\left(1 + \frac{\alpha_1\alpha_2}{16}\right)$, α_1 and α_2 being the initial and final small angular amplitudes in radians.

(b) The effect of the buoyancy of the air is to increase the period. The downward force on the bob is less than it would be *in vacuo* and the mass is unchanged.

(c) The air carried with the pendulum increases the period, since the moving mass is slightly greater than that of the sphere.

Method of Coincidences.—The timing of the pendulum can be

carried out over a period of several hours without counting the oscillations by the method of coincidences. This method is useful when the period of the pendulum is nearly a whole number of seconds.

The pendulum is placed in front of a standard pendulum clock. The suspending wire of the pendulum and the pointer attached to the clock bob are viewed through a telescope. In the rest position of the pendulum and clock these should be seen on the vertical cross wire of the telescope. The clock will have a period of 2 secs., the Borda pendulum a period of, say, slightly over 4 secs. As the pendulums swing they will both at some instant pass the cross-wire of the telescope together. When they next pass the cross-wire in the same direction the clock pointer will be slightly ahead. At the next transit of the clock pointer it will have a greater lead and so on. After a number of transits the lead will appear to diminish (the Borda wire will now be leading) until the pointer and wire are again very nearly coincident at a transit. The interval between successive coincidences is determined by noting the time by the standard clock for a number of coincidences. The interval cannot be determined accurately by observation of only a few coincidences, but it should be noted that an error of 1% in the coincidence interval produces a very small error in the period if the interval is greater than a hundred times the period.

During the coincidence interval a pendulum with a period of nearly 2 secs. would make one more vibration (or one less) than the clock pendulum. In the case of the Borda pendulum of period a little greater than 4 secs., the clock will make $2n + 2$ half vibrations, whilst the pendulum makes n half vibrations, in the coincidence interval of $(2n + 2)$ secs.

The half period of the pendulum is then

$$t = \frac{2n + 2}{n} = 2\left(1 + \frac{1}{n}\right) \text{ secs.}$$

and $$\delta t = -\frac{2}{n^2}.\delta n.$$

Owing to the fact that the wire and clock pointer may appear coincident during several successive transits, there may be some uncertainty as to the coincidence interval. A small error δn in n produces a percentage error in the coincidence interval of $\frac{2.\delta n}{2n + 2}.100.$

The corresponding percentage error in t

$$= \frac{\delta t}{t}.100 = -\frac{2}{n^2}.\delta n.\frac{n}{2n+2}.100$$
$$= -\frac{2\delta n}{2n+2}.100.\frac{1}{n}.$$

The percentage error in t is only $\frac{1}{n}$ of the percentage error in the coincidence interval. The closer the period of the pendulum to a whole number of seconds, the larger is n and the more accurate is the determination of t.

3. Kater's Pendulum

In 1817 Kater made use of the interchangeability of the centres of suspension and oscillation in the compound pendulum. In its simplest form Kater's pendulum consists of a metal rod carrying a heavy bob at one end and having two knife-edges A and B, B being fixed and A movable along the rod, Fig. 30 (a). The knife-edge A is adjusted until the periods about A and B are equal. AB is then the length of the equivalent simple pendulum and $g = \frac{4\pi^2}{T^2}.AB$.

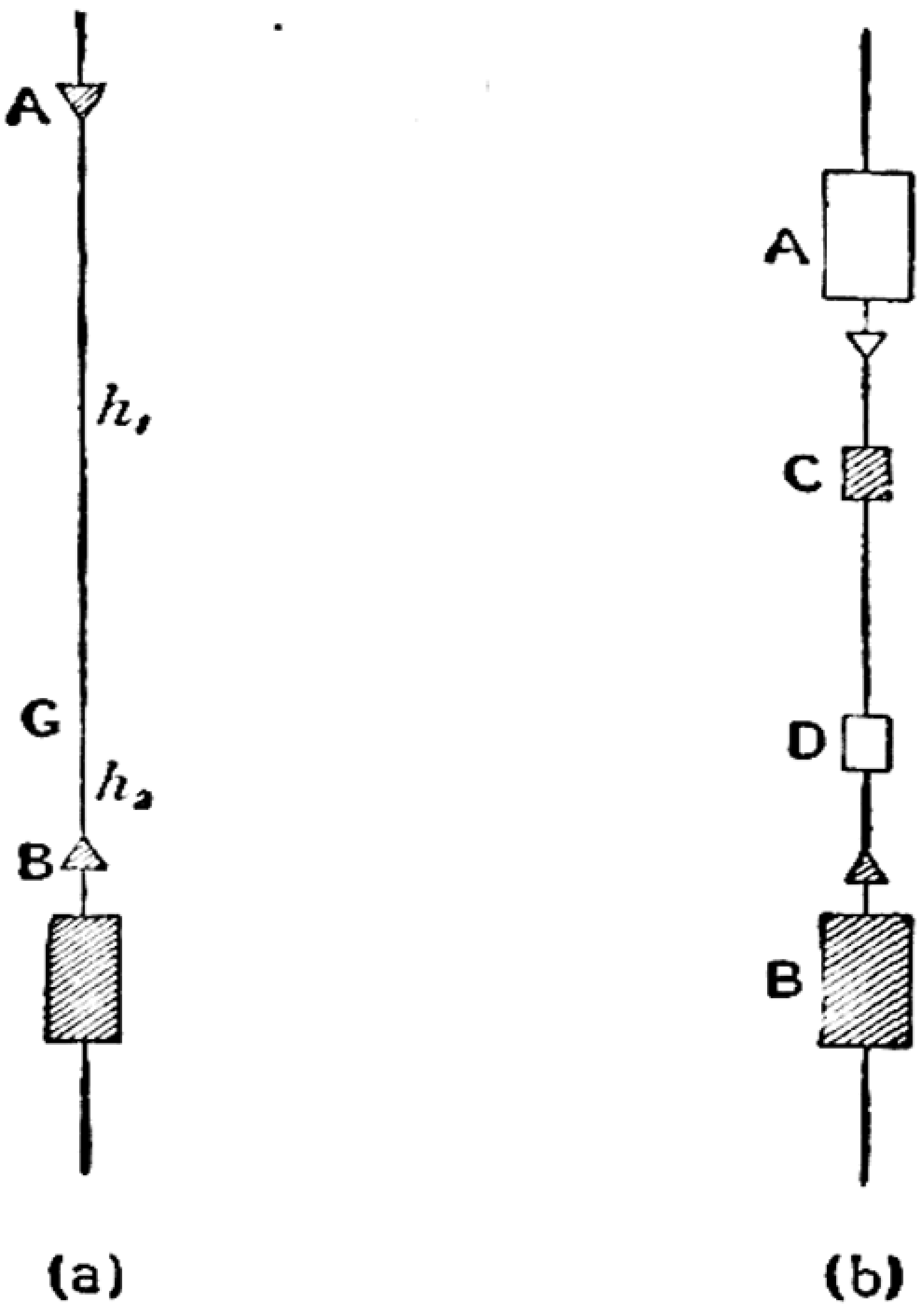

FIG. 30.

In practice it is not necessary to adjust A for exact equality of periods. It is sufficient if the periods are very nearly equal. We have, in this case, using our former notation for the compound pendulum,

$$\frac{gT_1^2}{4\pi^2} = \frac{K^2 + h_1^2}{h_1} \text{ and } \frac{gT_2^2}{4\pi^2} = \frac{K^2 + h_2^2}{h_2}.$$

Whence,
$$\frac{gT_1^2}{4\pi^2}.h_1 - h_1^2 = \frac{gT_2^2}{4\pi^2}.h_2 - h_2^2,$$

and
$$\frac{4\pi^2}{g} = \frac{T_1^2.h_1 - T_2^2.h_2}{h_1^2 - h_2^2}.$$

This may be written

$$\frac{8\pi^2}{g} = \frac{T_1^2 + T_2^2}{h_1 + h_2} + \frac{T_1^2 - T_2^2}{h_1 - h_2}.$$

In this T_1 and T_2 can be found accurately and $h_1 + h_2$ is AB the distance between the knife-edges. $h_1 - h_2$ can be found by balancing the pendulum on a wedge to find G, the centre of gravity. $h_1 - h_2$ need not be known accurately as the term containing this quantity has the small numerator $T_1^2 - T_2^2$.

Fig. 30 (*b*) shows another form of Kater's pendulum in which both knife-edges are fixed. B is the metal bob. A is of the same shape and size but made of wood. C is a metal weight which can slide along the rod. D is similar of wood. The pendulum is symmetrical about its centre. The times of vibration are adjusted to equality by moving the weight C, symmetry being preserved by moving D also, so that C and D are always at the same distance from the centre. With this type of pendulum the corrections for buoyancy and for the mass of air carried with the pendulum are eliminated.

4. The Invariable Half-second Pendulum

For determinations of g at a large number of places in survey work a shorter pendulum of period about 1 sec. with a single fixed knife-edge has been used. The time of vibration was measured at a place where g was accurately known and the values of g at other places deduced from the times of vibration of the pendulum at these places.

The Gravity Balance.—Of interest also in connection with survey work is the Gravity Balance. This instrument is capable of measuring quickly minute changes in the force of gravity. It consists of a horizontal torsion thread of quartz, carrying at its centre, and at right-angles to its length, a short wire which is slightly out of balance. The wire dips downwards on its heavy side and can be brought back to the horizontal by turning the torsion head at one end of the quartz thread. The angle through which the torsion head must be turned depends on the force of gravity at the place where the observation is made.

The instrument has been used in prospecting for oil, the boundaries of a subterranean lake being detected by the changes in gravity in the overlying area.

5. Variation of g. Effect of the Earth's Rotation

Owing to the spheroidal shape of the earth, g is less at the

equator than at the poles. It is calculated from observations at places of known latitude that the value of g at sea-level at the poles is 983·2 and at the equator 978·0, a difference of about $\frac{1}{2}$%. In latitude 45° the value is 980·6. (In ft.-sec. units these values are 32·26, 32·09 and 32·17.)

The difference between these values is due partly to the shape of the earth and partly to its rotation. The rotation of the earth has the further effect of causing the plumb-line to hang slightly out of the vertical in all latitudes except 0° and 90°.

To estimate the effect of the rotation let AB be the axis of the earth whose radius is R and centre O. Consider a mass of m gm. suspended by a thread at the equator. Let T be the tension of the thread and g the observed acceleration due to gravity. Then $\mathrm{T} = m.g$.

Let the pull of the earth on m be $m.\gamma$, γ being the acceleration due to gravity which would be experienced in the absence of rotation.

We have then, if ω is the angular velocity of the earth about AB,

$$m.\gamma - \mathrm{T} = m.\omega^2.\mathrm{R},$$

or

$$\gamma - g = \omega^2.\mathrm{R},$$

since m describes a circle of radius R.

$$\omega^2.\mathrm{R} = \left(\frac{2\pi}{24 \times 3600}\right)^2 \times 4000 \times 5280 \times 30{\cdot}5$$
$$= 3{\cdot}41.$$

The difference between the polar and equatorial $c.g.s.$ values of g is 5·2, of this 3·4 is due to rotation or centrifugal force.

Consider the mass m gm. suspended in latitude λ. The vertical at P is OP produced. The string will be at a small angle θ to OP produced, as in Fig. 31. The forces on m are T, $= mg$, where g is the observed acceleration due to gravity, and the earth's attraction $m.\gamma$ towards O. Since m

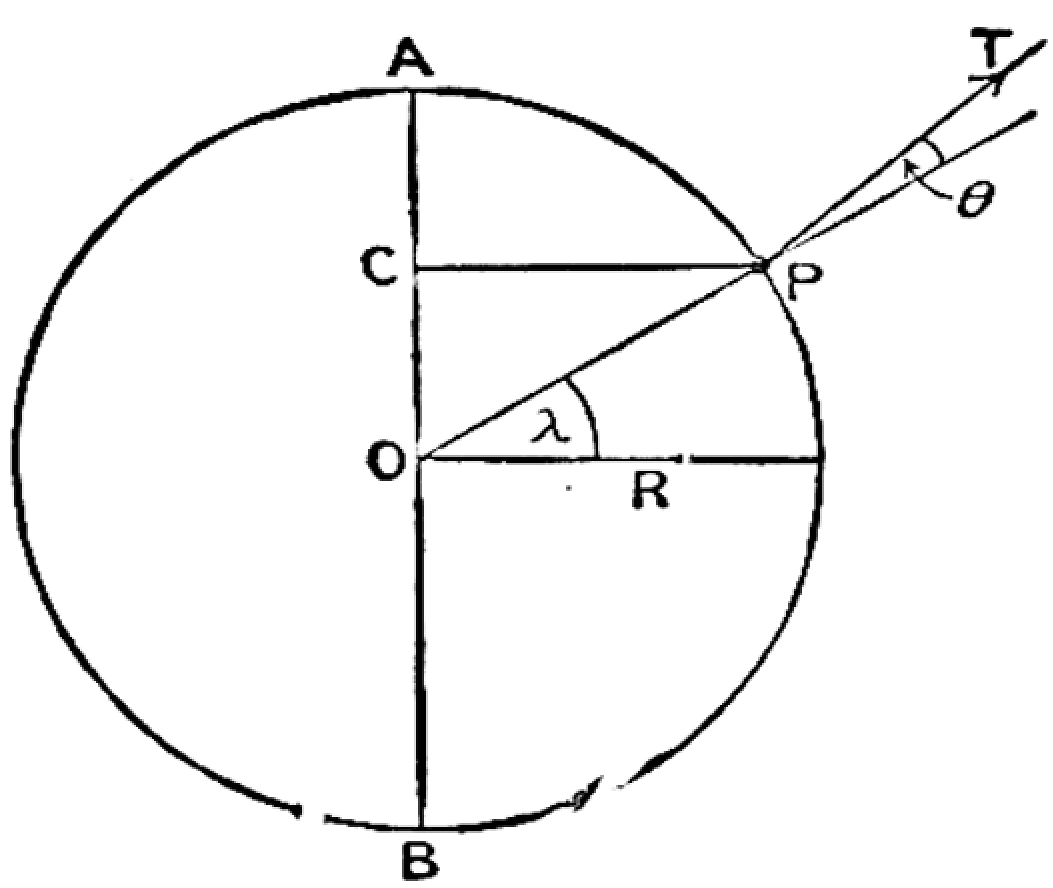

FIG. 31.

describes a circle centre C and radius PC, $= R\cos\lambda$, T and $m.\gamma$ must have a resultant $m.\omega^2.R\cos\lambda$ along PC.

Resolving along and perpendicular to PC we have

$$m.\gamma\cos\lambda - T\cos(\lambda+\theta) = m.\omega^2.R\cos\lambda$$

and $$m.\gamma\sin\lambda - T\sin(\lambda+\theta) = 0.$$

Or, since θ is small and $T = mg$,

$$\gamma\cos\lambda - g\cos\lambda + g.\theta.\sin\lambda = \omega^2 R\cos\lambda \quad . \quad . \quad (1)$$

and $$\gamma\sin\lambda - g\sin\lambda - g.\theta.\cos\lambda = 0 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Multiplying (1) by $\sin\lambda$ and (2) by $\cos\lambda$ and subtracting

$$g.\theta.\sin^2\lambda + g.\theta.\cos^2\lambda = \omega^2 R\sin\lambda.\cos\lambda$$

or $$\theta = \frac{\omega^2.R}{2g}.\sin 2\lambda.$$

Also from (2)

$$\gamma - g = g.\theta.\frac{\cos\lambda}{\sin\lambda} = \omega^2.R.\cos^2\lambda.$$

Thus $\gamma - g = 0$ at the poles and has a maximum value $\omega^2.R$ at the equator.

θ is greatest when $\lambda = 45°$, its value then is $\frac{3{\cdot}41}{2\times 980{\cdot}6}$ radians $= 6'$.

6. Kepler's Laws

From early times the motion of the planets has excited the interest of observers. Babylonian astronomers were able to predict their appearances with success. The Greek Ptolemy elaborated the system of circles and epicycles by which their apparent motions from the earth can be represented. The earth was assumed to be at rest at the centre of the universe.

Copernicus, looking for a simpler way of representing the planetary motions, developed the idea of the Greek Aristarchus that the sun was the centre of the universe and that the planets described circles about the sun as centre. This system was published in 1542.

In 1609, Kepler, as the result of a long study of the observations of the astronomer, Tycho Brahe, published the first two of his three laws.

(1) The planets describe ellipses having the sun at one focus.

(2) The line joining sun and planet sweeps out equal areas in equal times.

The third law was announced in 1618.

(3) The squares of the times of revolution of the planets are proportional to the cubes of their mean distances from the sun.

7. Newton's Law of Gravitation

Kepler's Laws are a summary of observations of the motions of the planets. They give a simple and fairly accurate description without offering any explanation. It was left to Newton to interpret Kepler's Laws in terms of the force of gravitation between sun and planet, and to give his new law of gravitation its general form.

Newton appears to have solved the problem of planetary motion about 1666 at the age of twenty-four but his discoveries were not announced until about twenty years later.

It is instructive to trace the steps by which Newton may have arrived at his final generalisation.

At the outset Kepler's third law suggests that the attraction between sun and planet is inversely proportional to the square of the distance between them. Suppose T is the period of revolution of a planet and ω its angular velocity about the sun; then for a circular orbit $\omega = \frac{2\pi}{\mathrm{T}}$, and the force on the planet towards the sun is $m.\omega^2.\mathrm{R}$, or $m\left(\frac{2\pi}{\mathrm{T}}\right)^2.\mathrm{R}$, where m is the mass of the planet and R the radius of the orbit. For different planets $\mathrm{T}^2 = k.\mathrm{R}^3$ (Kepler's third law), where k is some constant, the same for all the planets. Thus the force on the planet is proportional to $\frac{m}{\mathrm{R}^2}$.

Newton tested the assumption of the inverse square law in the case of the moon's motion. Let T be the moon's period of revolution about the earth (27 d. 7 hr. 43 min.); its acceleration towards the earth is $\left(\frac{2\pi}{\mathrm{T}}\right)^2.\mathrm{D}$, where D is its distance from the earth's centre.

If g is the acceleration due to gravity at the earth's surface, and a the radius of the earth, then, with the inverse square law $g.a^2$ should be equal to $\left(\frac{2\pi}{\mathrm{T}}\right)^2\mathrm{D}.\mathrm{D}^2$.

Now $\mathrm{D} = 60a$ roughly, and g should therefore be equal to $\frac{4\pi^2}{\mathrm{T}^2}.60^3.a$.

Expressing T in seconds and a in feet

$$\frac{4\pi^2}{T^2} . 60^3 . a = 32{\cdot}4,$$

with the rough values $T = 27\frac{1}{3}$ days and $a = 4{,}000$ miles.

Thus the inverse square law survived the test of the moon's motion.

It is assumed in the above reasoning that the gravitational force exerted by the earth acts as if the earth's mass were concentrated at its centre. Newton gave a geometrical proof of the truth of this assumption.

In the above preliminary investigation the orbits have been considered circular, whereas Kepler's first law describes them as elliptic. We have then to see what deductions are to be made from the facts embodied in Kepler's laws.

The second law is equivalent to the statement that the angular momentum of a planet about the sun is constant.

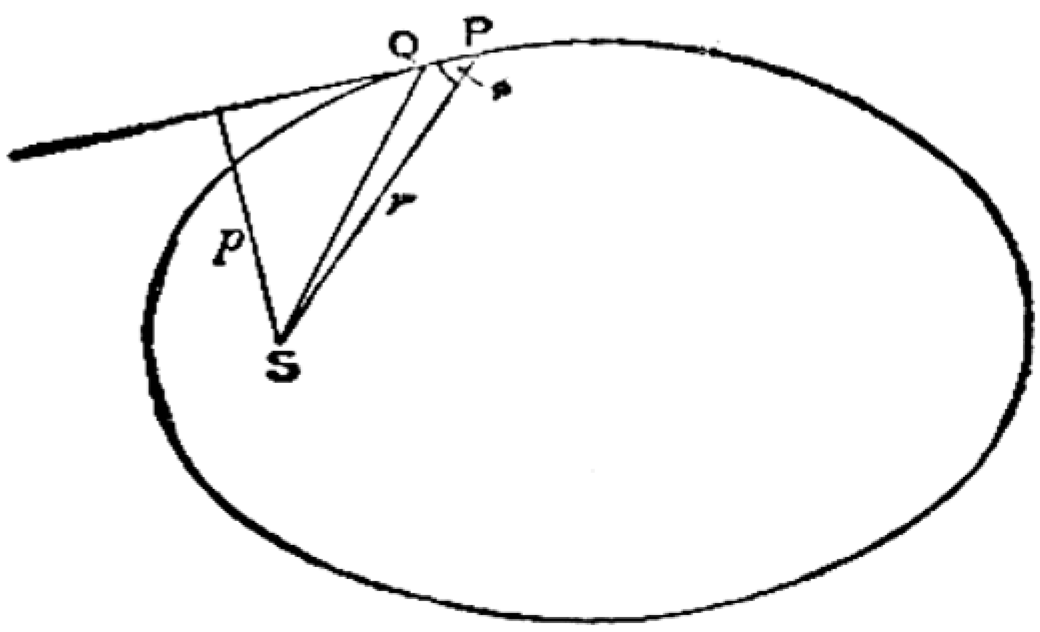

FIG. 32.

That this is so will be clear from Fig. 32. P, Q are the positions of the planet at the beginning and end of an interval δt. If v is the speed of the planet $PQ = v . \delta t$, and if A is the area traced out by SP from some fixed position, $\delta A = \Delta PSQ = \frac{1}{2} p . PQ$, p being the perpendicular from S to PQ.

Thus
$$\frac{dA}{dt} = \tfrac{1}{2} p . v.$$

Since equal areas are described by SP in equal times $\frac{dA}{dt}$ is constant, and so therefore is $mv . p$, the angular momentum about S, m being the planet's mass.

It is clear that there must be a force acting on the planet since its motion is not rectilinear, and also, since the angular momentum about the sun is constant, this force has no moment about the sun and must therefore be always directed towards the sun. This is the deduction from the second law.

From the first law it follows that the force must vary inversely as the square of the distance. In fact, this is the only law of

force consistent with an elliptic orbit having the sun at one focus.

To prove this let h be twice the area swept by SP, Fig. 32, in 1 sec., and let the force on the planet, of mass m, be denoted by F, $= f(r)$.

Then $$F \cos \phi = m . \frac{dv}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$\frac{dv}{dt}$ being the tangential acceleration of the planet.

Also $$p . v = h = \text{constant}.$$

Therefore $$p . \frac{dv}{dt} = - v . \frac{dp}{dt}.$$

With the usual notation (1) may be written

$$F . \frac{dr}{ds} = - \frac{mv}{p} . \frac{dp}{dt},$$

i.e., $$F = - \frac{mv}{p} . \frac{ds}{dt} . \frac{dp}{dr} = - \frac{mv^2}{p} . \frac{dp}{dr}.$$

The tangential-polar equation of the ellipse whose semi-axes are a and b is $\frac{b^2}{p^2} = \frac{2a}{r} - 1$.

Therefore $$\frac{dp}{dr} = \frac{p^3 . a}{r^2 b^2}$$

and $$F = - \frac{mh^2}{p^3} . \frac{p^3 a}{r^2 b^2} = - \frac{m}{r^2} . \frac{h^2 a}{b^2},$$

the negative sign indicating that the force is towards S.

If we write the force $F = \mu . \frac{m}{r^2}$, where $\mu = \frac{h^2 . a}{b^2}$, we have for the period T of the planet,

$$\frac{h}{2} . T = \pi . a . b, \text{ the area of the ellipse,}$$

i.e., $$T = \frac{2\pi ab}{h} = \frac{2\pi a^{\frac{3}{2}}}{\mu^{\frac{1}{2}}}$$

or $$T^2 = \frac{4\pi^2}{\mu} . a^3.$$

The meaning, then, of Kepler's third law is that μ has the same value for all the planets, and therefore that the force on a planet is proportional to its mass. Since the force is mutual it will also

be proportional to the mass of the other body. We have then

$$F = G.\frac{M.m}{r^2},$$

where G is a constant the same for all the planets and M is the mass of the sun.

It should be noticed that we have throughout considered the sun to be fixed. This is approximately the case since its mass is large compared with that of a planet. Strictly, both sun and planet describe ellipses having their centre of mass as a focus, the sun's ellipse being very small.

Newton then considered the attraction exerted by a body to be the resultant of the separate attractions of its individual particles, and stated his universal law of gravitation in the form :—

All particles of matter attract each other with a force proportional to the product of their masses and inversely proportional to the square of their distance apart,

or $$F = G.\frac{m.m'}{r^2},$$

where G is a universal constant whose value depends on the units in which mass, distance, and time are expressed.

It remains to show that with this law the attraction between a sphere and a particle is the same as if the sphere were replaced by a particle of the same mass placed at its centre. It follows then, since the attraction is mutual, that the force between two spheres will be the same as if they were both replaced by particles of their own masses at their centres.

8. Law of Attraction of a Sphere at Points outside and inside

(1) *Attraction of a Solid Sphere on a Particle Outside its Surface.*—Let a be the radius of the sphere of mass M. Its attraction on a particle of mass m distant R from its centre is $F = G.\frac{M.m}{R^2}$ if $R > a$.

This expression for the force is correct for a sphere of uniform density, and also for a sphere in which the density is not uniform provided it is the same at all points at equal distances from the centre. The stars, sun, and planets are approximately spheres of this type.

(2) *Attraction of a Solid Uniform Sphere on a Particle Inside its Surface.*—In this case $R < a$, and the force

$$F = G.\frac{M.m}{a^3}.R.$$

The force is directly proportional to the distance from the centre.

On the surface of the sphere $R = a$, and both expressions for F become $G.\frac{M.m}{a^2}$.

These results are proved by considering first the attraction of a thin spherical shell on a particle. It can be shown, by direct integration or by geometrical proof, that for points outside the shell the attraction is the same as if the mass of the shell were concentrated at its centre, and that for points inside it the resultant attraction is nil. Since a sphere whose density is uniform or symmetrical about its centre can be supposed built up of a large number of thin concentric spherical shells of uniform density the result (1) follows at once.

For a particle of mass m inside a uniform sphere at a distance R from the centre we notice that the attraction of those shells whose radii are greater than R will be nil. If ρ is the density of the sphere the attraction on the particle is

$$\frac{G.\frac{4}{3}\pi R^3.\rho.m}{R^2} = G.\frac{M.m}{a^3}.R,$$

since $M = \frac{4}{3}\pi a^3.\rho$.

It is useful to notice that since the law of gravitational force, $F = G.\frac{m.m'}{r^2}$, has the same form as the force between two electric charges, $F = \frac{1}{K}.\frac{e.e'}{r^2}$, many of the theorems of Electrostatics apply also to Gravitation. There exist, for example, in the neighbourhood of matter, a gravitational field and a gravitational potential. The potential at a point in the field is defined as the work done against the force of gravity in taking unit mass from the point to infinity. At a distance r from a particle or sphere of mass m the potential will be $G.\frac{m}{r}$ (as in Electrostatics). An electric charge placed on an isolated conducting sphere distributes itself uniformly over the surface and will correspond to a uniform thin spherical shell of attracting matter. In the case of the electric charge on a conducting sphere we know that the potential and electric intensity due to it at points outside the sphere are the same as if the charge were concentrated at its

centre, and that the intensity at all points inside the sphere is nil. A uniform thin spherical shell of matter will have similar properties.

9. Experimental Determination of G, the Gravitation Constant

The symbol G is used to denote the value of the gravitation constant when the masses, distance and force are in c.g.s. units. To determine G it is necessary to measure the force between two given masses (spheres in laboratory experiments) for a known distance between their centres. For masses of convenient laboratory size the force to be measured is only a fraction of a dyne.

A knowledge of G leads to a determination of the earth's mass and mean density. The early attempts to measure G are described as determinations of the earth's mean density. The experiments have been of three types:

(1) Mountain Methods.
(2) Torsion Balance Methods.
(3) Common Balance Methods.

(1) *Mountain Methods.*—In the years 1774–1776, Maskelyne made careful experiments on Mt. Schiehallion, in Perthshire. The general idea is illustrated in Fig. 33. Observations were made on the North and South sides of the mountain, of the angular distances from the zenith (as given by a plumb-line) of a

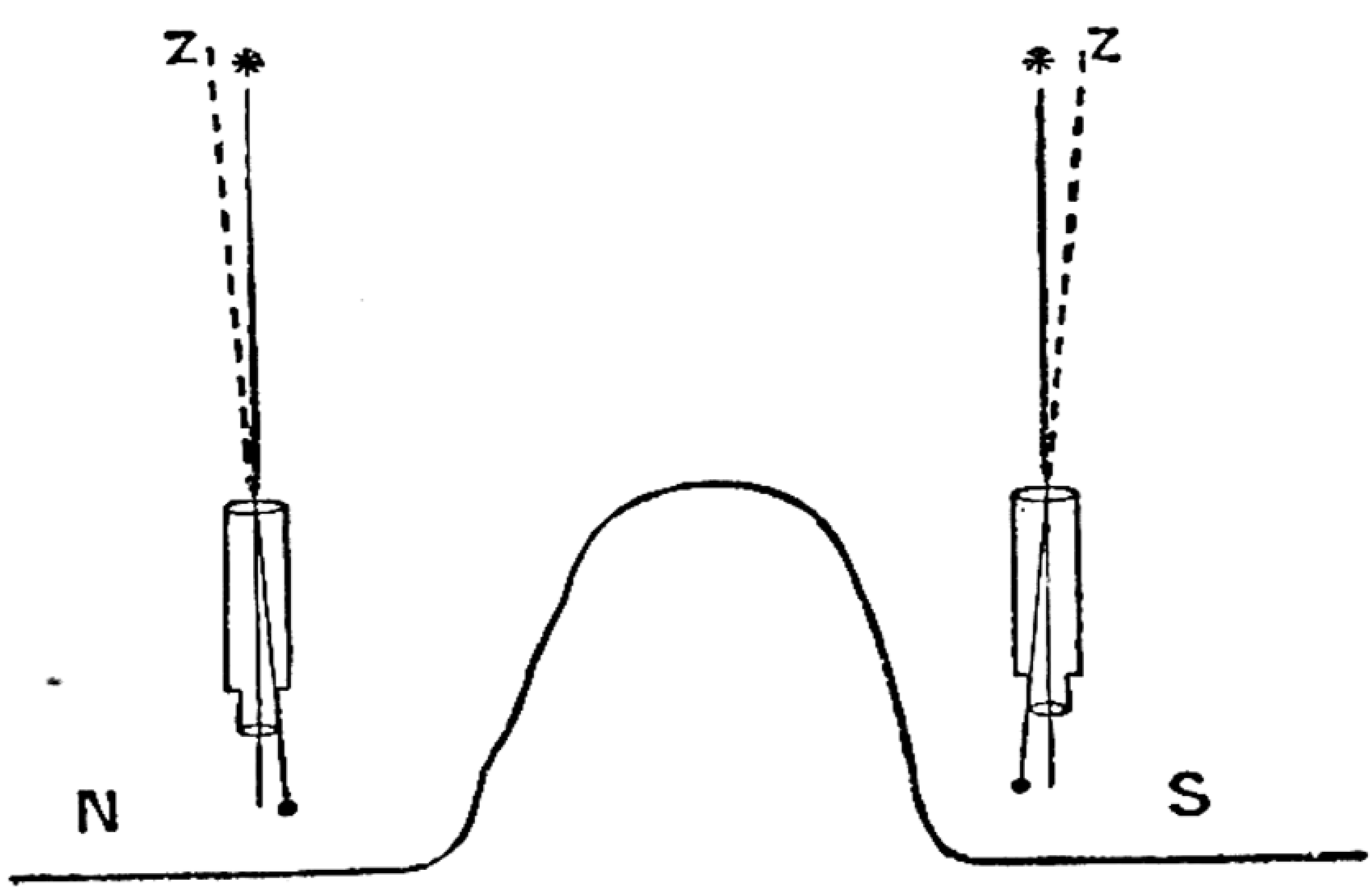

FIG. 33.

number of stars as they passed the meridian nearly overhead. The apparent change θ in the direction of a particular star when the telescope is moved from the North side to the South side of the mountain is due partly to the deflection of the plumb-line caused by the mountain and partly to the change in latitude. In the absence of the mountain the plumb-line would point to the centre of the earth, and an apparent change in the direction of the star would be observed equal to the difference in latitude $\delta\lambda$ of the two stations. The deflection of the plumb-line caused by the mountain is, therefore, $\frac{1}{2}(\theta - \delta\lambda)$. In the experiment θ was 55″ and $\delta\lambda$ 43″.

A careful survey of the mountain was made to determine its shape and volume. Its mean density was estimated from its geological formation. Its mass and effective distance from the plumb-line could then be calculated.

Here we have all the data required for the determination of G, the mass and the mean density of the earth. The mean density obtained was nearly 5 gm. per c.c.

Attempts to measure G have also been made by timing the vibrations of a pendulum at the top of a mountain and so finding g. At the top of the mountain g will be less than at the foot owing to the increased distance from the centre of the earth. The attraction of the mountain, however, causes an increase in g at the summit, and the value obtained by experiment is greater than that calculated from the height of the mountain. With a survey of the mountain the values of g at the base and on the summit enable G to be calculated.

Owing to the difficulty of estimating the mean density of a mountain these methods can give only an approximate value for G.

Example.—Let g and g' be the values of the acceleration due to gravity at the base and summit of a mountain of mass m and height h. Let d be the distance from the summit of the centre of attraction of the mountain. (This will not be the centre of mass.) Let R be the radius of the earth.

Then
$$g' = g \cdot \frac{\mathrm{R}^2}{(\mathrm{R}+h)^2} + \mathrm{G} \cdot \frac{m \cdot 1}{d^2}.$$

Whence G.

To obtain the mean density, Δ, of the earth, mass M, we notice that the attraction of the earth on 1 gm. at its surface is g dynes.

Therefore $$g = G.\frac{M.1}{R^2} = G.\frac{4\pi R^3.\Delta}{3R^2},$$

and $$\Delta = \frac{3g}{4\pi.G.R}.$$

(2) *Torsion Balance Methods.*—In 1798 Cavendish made the first accurate determination of G by measuring the attraction between lead spheres. A torsion rod AA, 6 ft. long, was supported by a fine torsion wire of copper about 40 in. long. The lead spheres, *mm* about 2 in. in diameter, were suspended from the ends of the rod by short lengths of thin wire. The torsion wire, torsion rod and spheres *mm* were enclosed in a narrow wooden case (not shown in Fig. 34) supported from the floor of a chamber built round the whole apparatus. Two large spheres

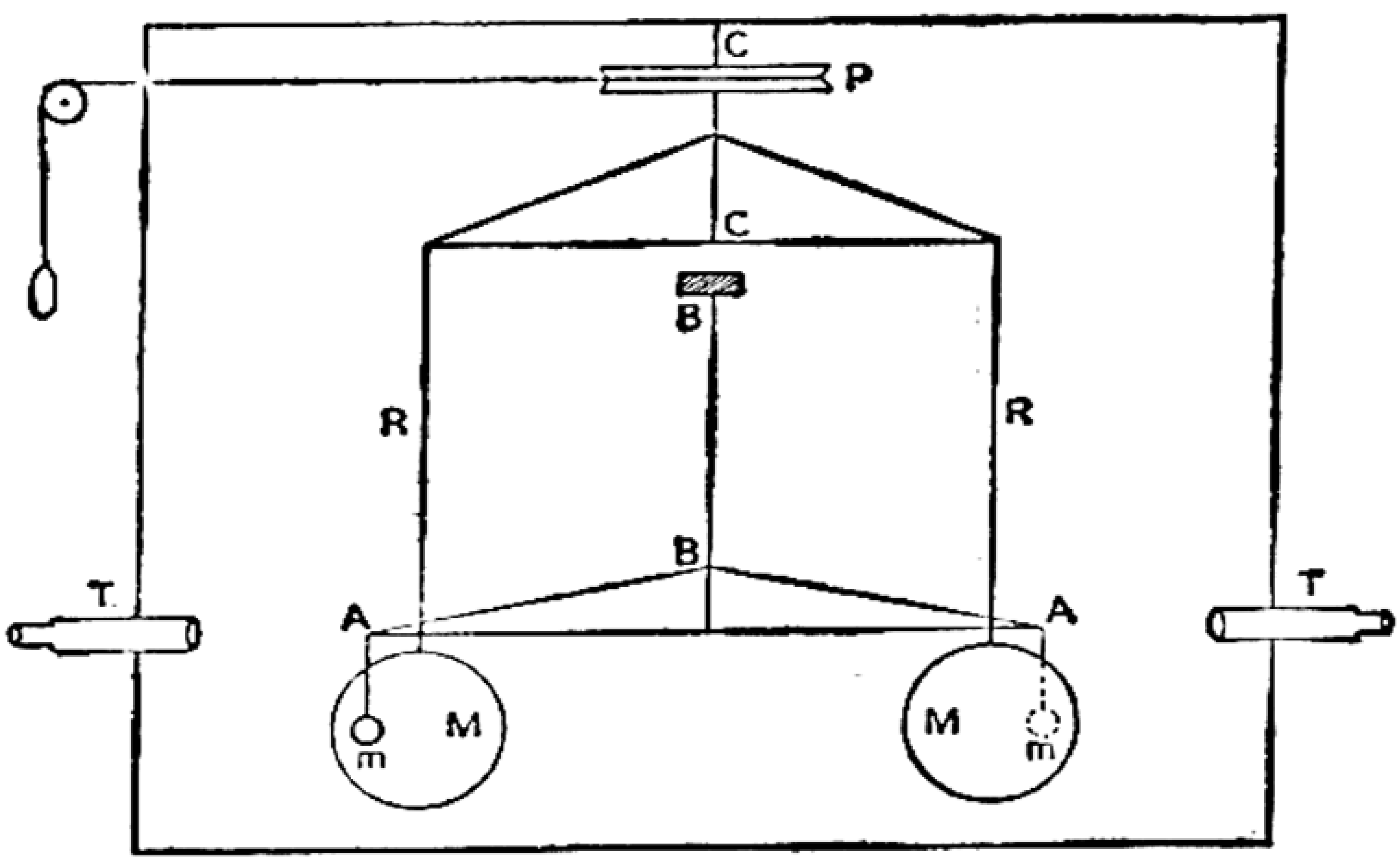

FIG. 34.

MM, about 1 ft. in diameter, also of lead, hung on rods RR so that their centres were on the same level as those of *mm*. By means of the rope and pulley P these spheres could be turned about the axis CC so as to be close to *mm* on either side. Verniers were fixed on the ends of the torsion rod and moved close to scales attached to the narrow case. The scales and verniers could be observed by means of telescopes from outside the chamber. In spite of the precautions taken to avoid convection currents inside the chamber the torsion rod was never at rest, and it was necessary to find the centre of swing on the scale with the masses MM first on one side and then on the other. The distance between these two centres of swing is twice the

deflection produced by the attractions of the spheres. The masses MM were brought up against fixed stops where the distance of their centres from the zero position of the spheres mm was known.

The torsion constant of the wire, *i.e.*, the couple required to twist it through 1 radian, was found by measuring the time of oscillation of the rod and masses mm when the spheres MM were removed. If c is the constant of the wire $T = 2\pi\sqrt{\frac{I}{c}}$, T was measured and I, the moment of inertia of the rod and spheres mm, was calculated from their masses and dimensions. Let the rod AA have length $2a$ and let d be the distance between the centres of the spheres. The couple on the rod is $G.\frac{M.m}{d^2}.2a$.

If θ is the deflection produced

$$c.\theta = G.\frac{M.m.2a}{d^2}$$

from which G is found in terms of measured quantities.

Corrections were applied by Cavendish for the attraction of the distant mass M, for the attractions on the rod AA and for the influence of the rods RR. The attraction of the narrow case was found to be negligible for such deviations of the torsion rod from its central position as were obtained in the experiment.

Cavendish's mean density, the result of about thirty experiments, was 5·45 gm. per c.c. The mean of the best later determinations is 5·52.

Boys' Modification of Cavendish's Method.—In 1895 Professor Boys repeated the Cavendish experiment, using a very fine quartz thread as torsion wire. This enabled him to reduce greatly the dimensions of the apparatus and so avoid the disturbances caused by convection currents. The small masses mm were of gold, one pair used having a diameter of 0·25 in. The torsion rod A was a strip of mirror 0·9 in. long. The torsion thread BB was 17 in. long. The deflection of the torsion rod was measured by observing with a telescope the reflection in the mirror of a scale 23 ft. away. In the experiments the deflection was about 1°.

The large spheres MM, of lead 4·5 in. in diameter, were suspended from the top of the outer cylindrical case. This cover CC could be rotated about the central axis. The spheres MM were brought into the position giving maximum deflection first

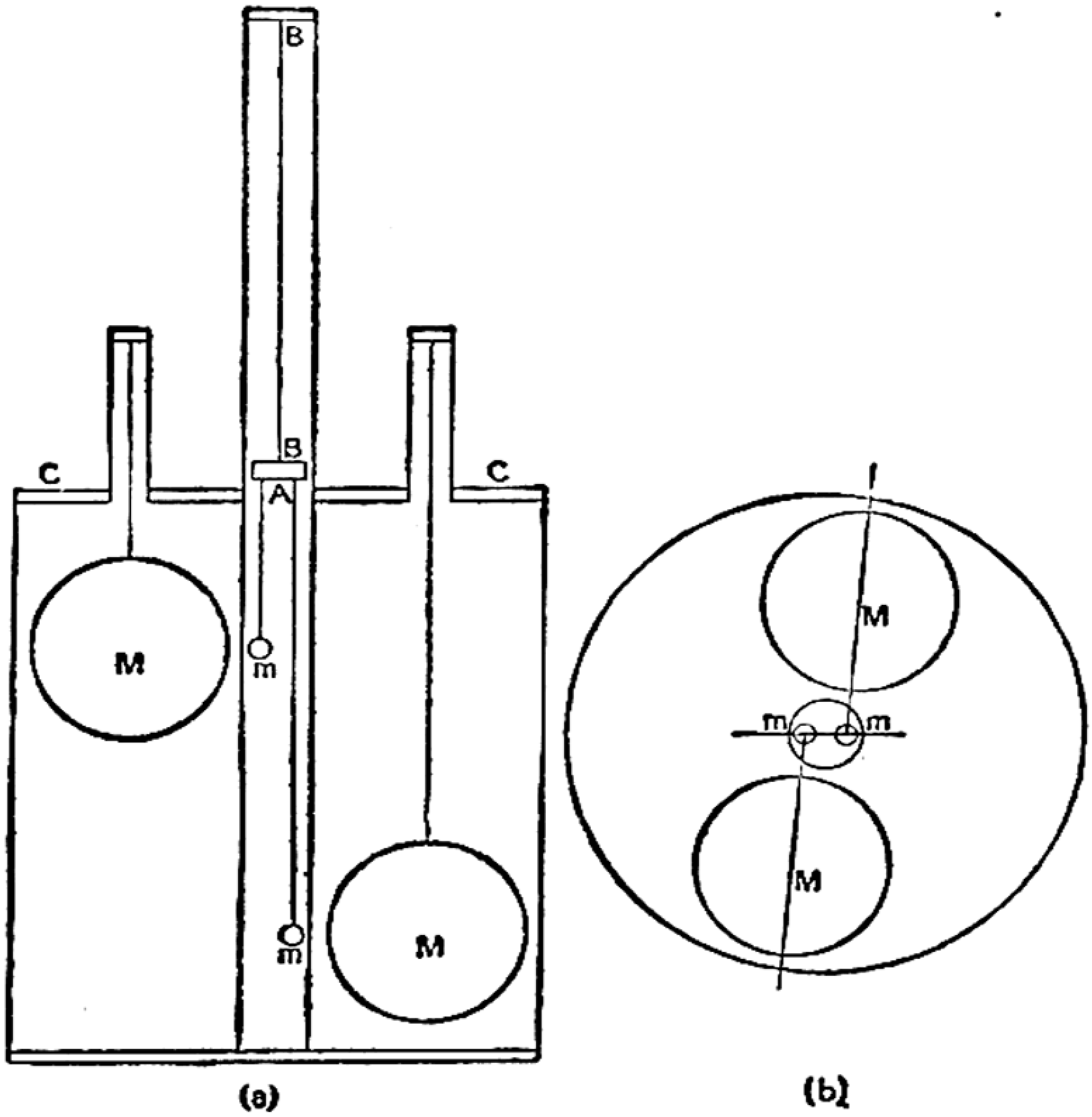

FIG. 35.

on one side, then on the other. With this short torsion rod it was necessary to suspend the spheres at different levels or the attraction of each of the large masses would have been nearly neutralised by the other. The apparatus was constructed with great care and its dimensions accurately measured.

The position of the spheres for maximum deflection is shown approximately in Fig. 35 (*b*). The line of attraction M*m* is not perpendicular to *mm*, but is inclined to it at an angle of about 100°, depending on the dimensions of the apparatus. As the angle M*mm* is decreased from 180° to 90° the distance M*m* increases slightly and the attraction becomes smaller on this account, though its turning moment is increasing as perpendicularity is approached. The quartz threads attaching *mm* to A will not be exactly vertical or the rod A would not be deflected. The correction required on this account to the distance between the centres of the spheres would, however, be less than 1 in 10^6.

The final result was $\Delta = 5{\cdot}527$

and $G = 6{\cdot}658 \times 10^{-8}$.

For calculations it is usual to take

$$G = 6{\cdot}66 \times 10^{-8}.$$

(3) *Common Balance Methods.*—A few observers have made use of the common balance to find the attraction between

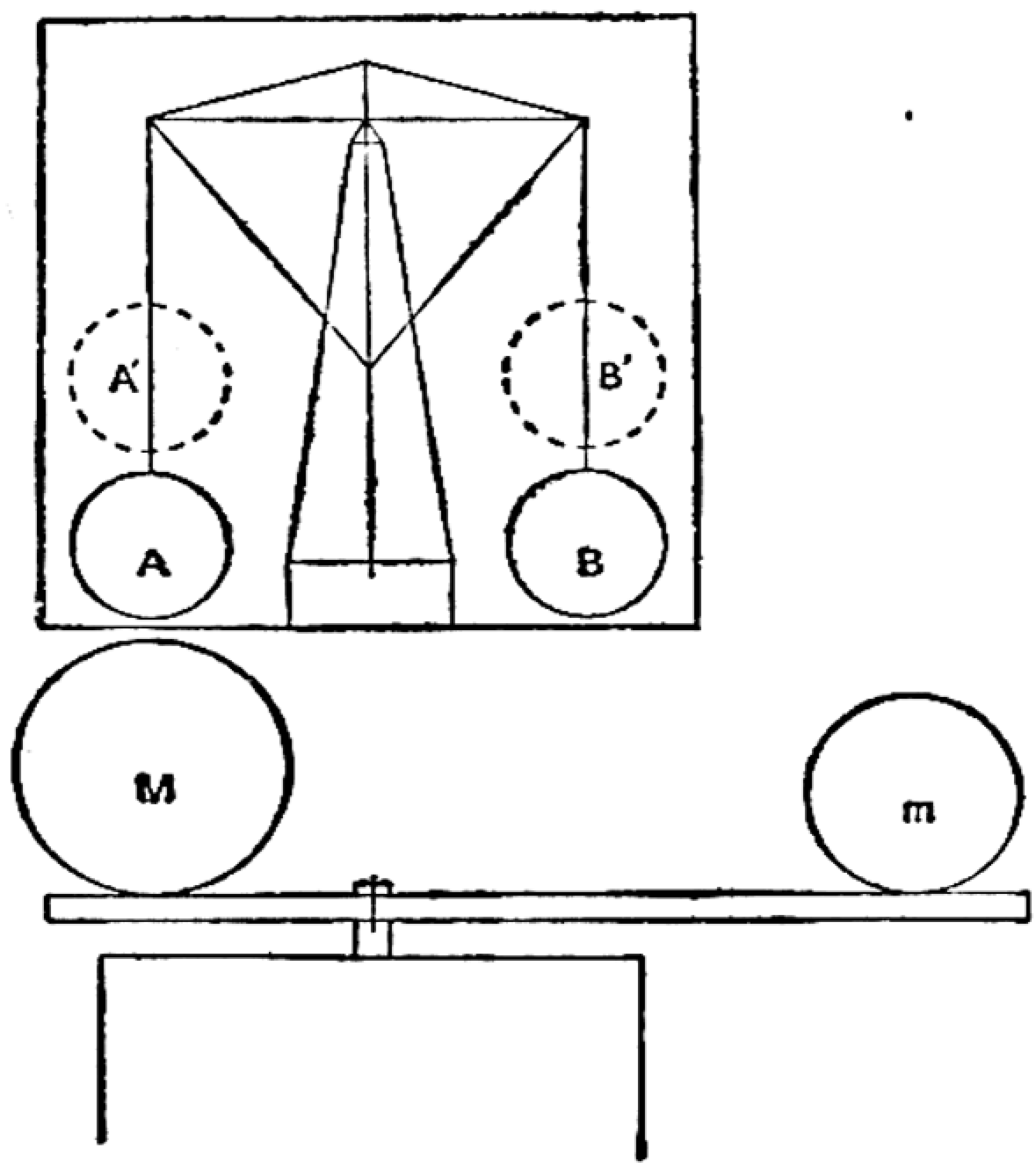

FIG. 36.

masses of known size. Professor Poynting's determination of G, in 1878, is typical of these experiments. Two equal lead spheres, 50 lb. each, were suspended from the arms of a strongly made balance. A turntable under the balance carried a lead sphere M of mass 350 lb. and a smaller mass m as counterpoise for M. The sphere M could thus be placed directly under A or B. The deflection of the beam on moving M from one position to the other was observed. A centigram rider was then moved along the balance beam to give the same deflection. To eliminate the effect of the attraction of M on the beam of the balance the experiment was repeated with A and B in the higher positions A′, B′. The attraction on the beam remained the same and, by subtraction of the two results, could be eliminated.

The final result was

$$G = 6{\cdot}698 \times 10^{-8}$$
$$\Delta = 5{\cdot}493.$$

10. Density and Mass of the Earth

The mean of the most reliable determinations of Δ is 5·52 gm. per c.c.

The mean density of the surface layer is 2·65 gm. per c.c.

In shape the earth is a spheroid of revolution, the equatorial radius being 6,378,300 metres, the polar radius 6,356,850 metres.

The volume may be calculated from the formula $V = \frac{4}{3}\pi . b^2 . a$, where a is the polar radius and b the equatorial radius.

The mass of the earth may then be obtained by multiplying the volume by the mean density 5·52 gm. per c.c.

The result is $5{\cdot}98 \times 10^{27}$ gms.

11. Some Astronomical Calculations

In the following examples, E, M, S denote the masses of the earth, moon and sun respectively. In this type of calculation the candidate in examinations is occasionally expected to supply approximate data, such as the earth's radius, 4,000 miles; the moon's period of revolution, 27·3 or 28 days; the length of our year, 365¼ days; and the value of g.

Example 1.—Calculate the moon's distance.

We have the period of revolution T = 27·3 days, and the moon's angular velocity about the earth's centre

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{27{\cdot}3 \times 24 \times 3600} \text{ radians per sec.}$$

Also

$$G . \frac{E . M}{D^2} = M . \omega^2 . D,$$

D being the moon's distance in cms.,

and

$$G . \frac{E \times 1}{R^2} = 981 \text{ dynes},$$

the attraction of the earth on 1 gm. at its surface, R being the earth's radius in cms.

Dividing the above equations,

$$\frac{D^2}{R^2}=\frac{981}{\omega^2 . D}$$

or
$$D^3=\frac{981\ R^2}{\omega^2}$$

$$=\frac{981\times(4000\times5280\times30{\cdot}5)^2\times(27{\cdot}3\times24\times3600)^2}{4\pi^2}.$$

Whence $D=3{\cdot}86\times10^{10}$ cms.

Example 2.—Find the period of a small satellite close to the earth's surface.

Let m be the mass of the satellite, then, if ω is its angular velocity,

$$G.\frac{E.m}{R^2}=m.\omega^2.R$$

and
$$G.\frac{E\times1}{R^2}=981.$$

Therefore
$$981=\omega^2.R$$

and
$$T=\frac{2\pi}{\omega}=2\pi\sqrt{\frac{R}{981}}$$

$$=1\text{ hr. }24\text{ mins.}$$

Example 3.—Find the sun's mass in terms of that of the earth.

Let the periods of the moon about the earth and the earth about the sun be T_1 and T_2 respectively, and the radii of their orbits, assumed circular, be D_1 and D_2.

Then
$$G.\frac{S.E}{D_2{}^2}=E\left(\frac{2\pi}{T_2}\right)^2.D_2$$

and
$$G.\frac{E.M}{D_1{}^2}=M\left(\frac{2\pi}{T_1}\right)^2.D_1.$$

Dividing,
$$\frac{S}{E}=\left(\frac{D_2}{D_1}\right)^3.\left(\frac{T_1}{T_2}\right)^2.$$

If a planet has a satellite whose distance from, and period about, the planet are known from astronomical observations, the mass of the planet can be found as above in terms of that of the sun.

This method of calculating the ratio of the masses of the sun and the earth originated with Newton, but the actual mass of the sun could not be found until G had been measured.

Given G, S can readily be found from the first of the above equations in terms of the length of the year and the sun's mean distance from the earth.

12. The Velocity of Escape

Consider a mass m at a distance x (greater than the radius) from the earth's centre. The attraction on it is $\frac{G.E.m}{x^2}$. The work required to pull the mass m through a further distance δx is $\frac{G.E.m}{x^2}.\delta x$, and the work required to move the mass from the earth's surface to infinity is, therefore,

$$\int_R^\infty \frac{G.E.m}{x^2}.dx = G.E.m\left[-\frac{1}{x}\right]_R^\infty = \frac{G.E.m}{R}.$$

If the particle were allowed to fall towards the earth from infinity, its velocity on reaching the surface would be given by

$$\frac{1}{2}m.v^2 = \frac{G.E.m}{R},$$

i.e.,

$$v^2 = \frac{2.G.E}{R}.$$

In the same way, if the particle be projected away from the earth with a velocity equal to or greater than $\sqrt{\frac{2G.E}{R}}$, it will reach infinity and escape from the earth's control.

The above statements are true also if R is any finite distance from the earth's centre greater than the radius.

Fig. 37 shows the different paths of a particle projected with

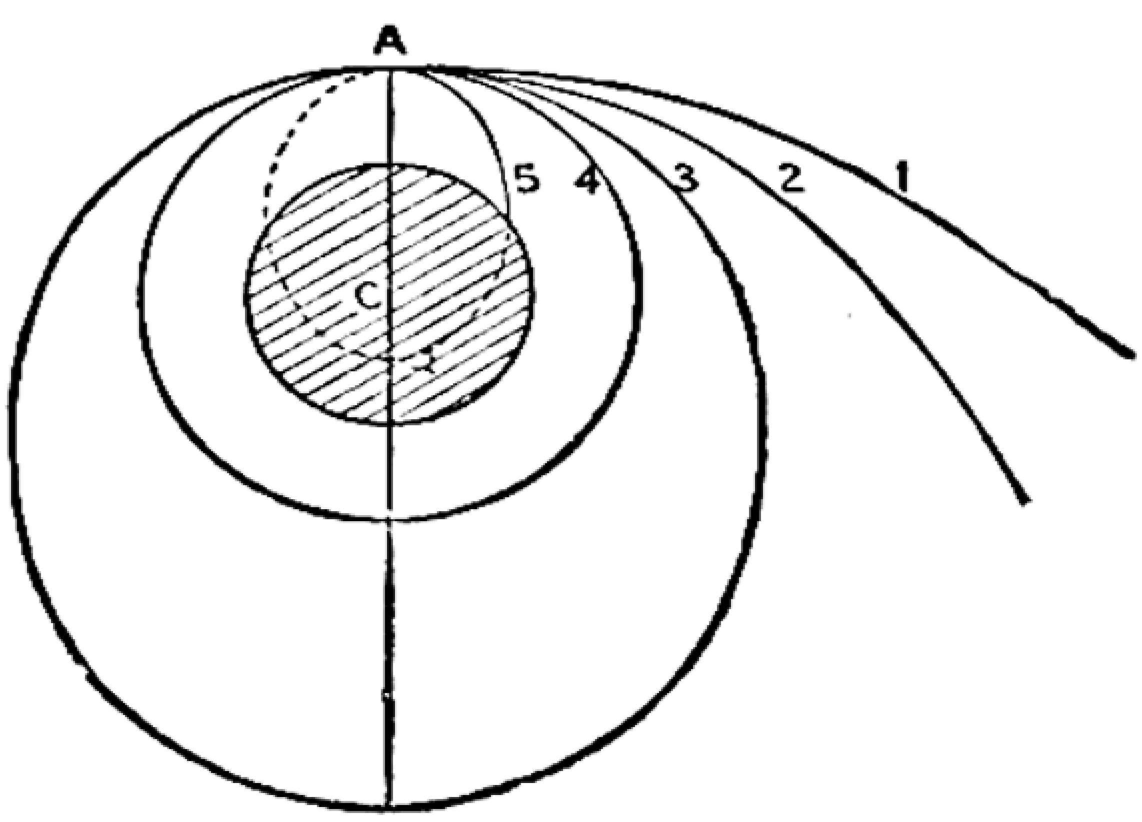

FIG. 37.

various velocities from A at right-angles to the radius through A. The centre C of the earth is a focus of all the paths. The earth is shaded and CA is denoted by R.

1. Hyperbola, $V^2 > \frac{2GE}{R}$.
2. Parabola, $V^2 = \frac{2GE}{R}$.
3. Ellipse, $V^2 < \frac{2GE}{R}$, but $> \frac{GE}{R}$.
4. Circle, $V^2 = \frac{G.E}{R}$.
5. Ellipse, $V^2 < \frac{GE}{R}$.

Path 5 is that of bodies projected with ordinary low velocities. If the point A is close to the earth's surface, path 5 is indistinguishable from the parabolic trajectory which is obtained in Mechanics by neglecting the variation of g with height.

It should be noticed that the velocity for escape is $\sqrt{2}$ times that for a circular orbit. If the moon's speed in its orbit were suddenly increased by 50% its path would become hyperbolic and it would depart for ever.

For a body near the earth's surface the velocity of escape $\sqrt{\frac{2G.E}{R}} = 11{\cdot}2\ 10^5$ cm. per sec.

For the moon's surface the velocity of escape is $2{\cdot}38\ 10^5$ cm. per sec.

The root-mean-square velocity of the molecules at 0° C. of hydrogen is $1{\cdot}84\ 10^5$ cm. per sec. For water vapour the figure is $\cdot71\ 10^5$, for nitrogen $\cdot49\ 10^5$, for oxygen $\cdot46\ 10^5$, and for carbon dioxide $\cdot39\ 10^5$. For 100° C. these figures would be about 17% greater. If the average velocity of the molecules of a gas is as high as $\frac{1}{5}$ of the velocity of escape, the faster molecules will escape from the upper regions of the atmosphere and in time that gas will disappear from the atmosphere. These figures explain the absence of an atmosphere on the moon and the rarity of the light gases, hydrogen and helium, in the earth's atmosphere.

13. Relativity and Gravitation

Newton's 'force of gravitation' accounts for the motion of the heavenly bodies with wonderful accuracy, but no explanation

is given as to how the force arises. By means of this force one body is able to produce an effect on another over great distances without any time lag and with no apparent connection between the bodies. This difficulty has been resolved by Einstein in his 'General Theory of Relativity' (1915). According to this theory a planet describing its orbit about the sun is in the same condition as the body describing a straight line with uniform velocity in Newton's First Law of Motion. The curved orbit is the result of something analogous to curvature impressed upon the surrounding space-time by the presence of the sun's mass. In this modified space the path of an unconstrained body is no longer rectilinear. The conception of 'force' is dropped and the sun produces its effect on a planet by modifying the space-time in which the planet exists.

The theory has its origin in the assumption that natural laws should have the same mathematical form when expressed with reference to all systems of coordinates. The general law of gravitation which is deduced from this assumption by a long and difficult train of mathematical reasoning reduces for weak gravitational fields to Newton's Law as a first approximation. On making a closer approximation deviations from Newton's Law appear, but except in one case these deviations are too small to be perceived in the motions of the planets. According to Newton's theory the elliptical orbits of the planets should remain fixed in direction with reference to the fixed stars. According to Einstein's their major axes should rotate slowly. Except in the case of Mercury this rotation is too small to be measured. For Mercury the rotation has been measured. It amounts to 43″ in a century. Newton's theory is powerless to explain this motion, but it agrees almost exactly with that calculated from the General Theory of Relativity.

Confirmation of Einstein's theory has also been obtained in other directions.

It should be noticed, however, that Newton's Law is not demolished by Einstein's more general account of gravitation. Owing to its simple form and high degree of accuracy it still retains its usefulness for the purposes of calculation. As for Newton's Laws of Motion, based on the conception of force, it is sufficient to say that the system of Mechanics built on them has proved extremely successful in its practical applications, and that they will always serve as the best introduction to the study of Mechanics and Physics.

Examples

1. Describe and explain a method of determining the acceleration due to gravity from experiments on a body falling freely. Discuss the accuracy of the method you describe. (O. and C.)

2. Describe an accurate method of measuring 'g.' Has 'g' the same value at all points on the earth's surface ? (C. Schol.)

3. Explain and describe the method of coincidences for comparing the time of vibration of a pendulum with that of a standard pendulum. Why are the results obtained by this method more accurate when the times of vibrations of the pendulums are nearly equal than when they are far apart ? (C. Schol.)

4. A train travels along the equator with a uniform velocity of 60 m.p.h. Discuss the effect of this motion on the weight of a body in the train. Does it make any difference whether the train travels east to west or west to east ? (N.U.)

5. Describe fully one of the good methods for the measurement of the constant of gravitation. How can the mass of the earth be deduced when the value of the gravitation constant is known ? (N.U.)

6. Discuss Newton's Laws of Motion.

What experiments would you make with an Atwood machine in order to verify the Second Law ? (N.U.)

7. If you already knew the radius and the mean density of the earth, what experiment would you make in order to determine the gravitation constant ?

A pendulum which has a period of exactly 1 sec. at the earth's surface would, if taken to the surface of the moon, have a period of 2·22 secs. Compare the mean densities of the earth and moon if their radii are in the ratio 3·7 to 1. (N.U.)

8. Outline a method for determining the mass of the earth.

Assuming that a sphere of mass 40 kilos is attracted by a second sphere of mass 80 kilos, when their centres are 30 cm. apart, with a force equal to the weight of $\frac{1}{4}$ mg., calculate the constant of gravitation. (N.U.)

9. Give a brief survey of the evidence that led Newton to formulate his Law of Gravitation.

A small satellite revolves round a planet of mean density 10 gm. per c.c., the radius of the satellite orbit being very little greater than the radius of the planet. If G is $6{\cdot}66\ 10^{-8}$ c.g.s. units, calculate the time of revolution of the satellite. (N.U.)

10. Assuming that the moon describes a circular orbit of radius $3{\cdot}84\ 10^{8}$ metres in 27·3 days and that the outer satellite

of Mars describes a circular orbit of radius $2{\cdot}35\ 10^7$ metres in $1{\cdot}26$ days, find the ratio of the mass of Mars to the mass of the earth. (C. Schol.)

11. Describe Cavendish's method of finding G, and explain how the mass of the earth may be deduced from the results of the experiment.

In what respects do you consider the apparatus used by Boys superior to that of Cavendish ?

12. Assuming the earth's orbit to be circular and of radius $1{\cdot}5\ 10^{13}$ cm. and G to be $6{\cdot}66\ 10^{-8}$ c.g.s. units, calculate the mass of the sun. (C. Schol.)

13. Describe experiments which have been made to determine the mass of the earth. (C. Schol.)

14. Calculate the period of revolution of a small satellite which describes an orbit close to the surface of a spherical planet of unit density. ($G = 6{\cdot}66\ 10^{-8}$ c.g.s.) (C. Schol.)

15. Give a brief account of the laws of gravitation. What must be the least muzzle velocity of a shell fired from the earth's surface which will just escape from the earth ? (Neglect friction due to the air.)

$G = 7{\cdot}0\ 10^{-8}$ c.g.s. ; mass of earth $M = 5 \times 10^{27}$ gm. ; radius of earth $R = 6{,}700$ km. (C. Schol.)

16. Form an estimate of the work which must be done against the gravitational attraction between the molecules in evaporating a sphere of mass M and unit radius. (O. Schol.)

CHAPTER V

HYDROSTATICS

1. Initial Assumptions

HYDROSTATICS is concerned with the mechanics of liquids in a state of equilibrium.

We start with two assumptions based on the observed properties of liquids :—

(1) The compressibility of liquids is in general so small that for most calculations it is safe to assume incompressibility. In other words the density is independent of the pressure.

(2) A liquid at rest is incapable of exerting a tangential stress either on the surface of a solid with which it is in contact or on the surface of any portion of the liquid itself. If the liquid is in motion tangential stresses may exist and give rise to the property of viscosity, but when the motion ceases they appear to vanish. It may be assumed, then, that any force exerted by a liquid at rest on a surface is at all points normal to the surface.

2. Pressure in a Liquid

The force exerted by a liquid on a surface per unit area is called the pressure on the surface. Since the pressure varies in a liquid (*e.g.*, with the depth) the idea of pressure at a point is necessary. This is defined as the force or thrust on a small area surrounding the point divided by the area.

A pressure may be stated in gravitational units, gm. wt. per sq. cm., lb. wt. per sq. ft., lb. wt. per sq. in., or, in absolute units, dynes per sq. cm., poundals per sq. ft.

The pressure at a point in a liquid is the same in all directions.

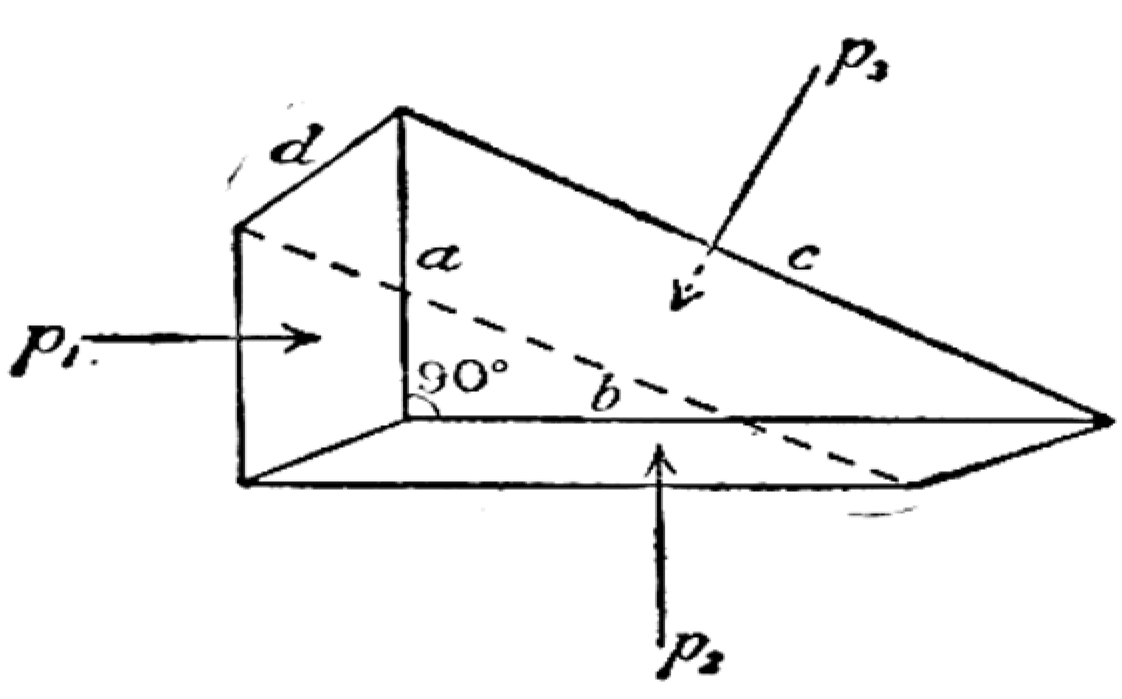

FIG. 38.

Consider the equilibrium of a portion of the liquid contained in a small triangular prism

(as shown in Fig. 38). Let the pressures on the faces of the prism be as shown, p_1, p_2, p_3. Let the edges be as marked a, b, c, d, edge a being vertical and the base horizontal. Let ρ be the density of the liquid, and α the angle between b and c.

Resolving horizontally and vertically we have

$$p_1 . ad = p_3 . cd . \sin\alpha = p_3 . ad,$$

i.e., $$p_1 = p_3.$$

Also $$p_2 . bd = p_3 . cd . \cos\alpha + \tfrac{1}{2} ab . d . \rho . g,$$

i.e., $$p_2 = p_3 + \tfrac{1}{2} a . \rho . g.$$

If the pyramid is small enough $\frac{1}{2} a . \rho . g$ is negligible ; p_1, p_2, p_3 become pressures at a point in different directions, and $p_1 = p_2 = p_3$.

The pressure at a point in a liquid at depth 'h' below the surface. If A is the pressure on the surface of the liquid it is shown in most elementary text-books of Physics that the pressure p at a depth h in a liquid of density ρ is given by $p = \mathrm{A} + h\rho g$ in absolute units. Thus the applied pressure A is transmitted to all parts of the liquid and is superposed on the 'hydrostatic pressure,' $h\rho g$, which is proportional to the depth below the surface.

Further, the pressure in a liquid at rest is the same at all points on the same level. Here it should be noted that the points must be in the same liquid and it must be possible to pass from one to the other without leaving the liquid.

3. Archimedes' Principle

A solid partly or wholly immersed in a liquid is exposed over the surface in contact with the liquid to a system of normal pressures. If we imagine the body to be removed and its place to be taken by a further quantity of the liquid this extra liquid will be exposed over its surface to the same system of pressures as acted on the solid, and it will be in equilibrium. Thus its weight is supported by the pressures over its surface. This system of pressures must therefore have a resultant which is a single force acting vertically upwards through the centre of gravity of the displaced liquid and equal to the weight of the displaced liquid.

The principle may be stated thus :—

The upthrust on a body, wholly or partially immersed in a liquid, is equal to the weight of liquid displaced and acts vertically upwards through the centre of gravity of the displaced liquid.

It applies equally, of course, to a body immersed in a gas.

The upward resultant of the system of pressures (called the

upthrust, or force of buoyancy) on an immersed body arises from the fact that the pressure increases with the depth in the liquid, so that the more deeply immersed parts of the body, on which the pressures tend to be upwards, are the parts experiencing the greater pressures.

When a body floats in a liquid, either wholly or partially immersed, it follows that the weight of liquid displaced by it is equal to its own weight. If it is floating wholly immersed its mean density must be the same as that of the liquid.

In the case of a body floating partially immersed in a liquid we should strictly take into account the air displaced by the portion outside the liquid, and state that the weight of the body is equal to the sum of the weights of liquid and air displaced.

In most cases the weight of air displaced is only a small fraction of the weight of liquid displaced, and it is usually neglected.

4. The Buoyancy Correction in Weighing

In accurate weighing the buoyancy of the air cannot be neglected. Of course, if a piece of brass is weighed with brass weights, the weights and the piece of brass will have the same volume and displace the same weight of air, and so no correction will be needed. But when the density of the object weighed differs from that of the weights, as is usually the case, a correction is necessary to obtain the true weight.

Let W be the true weight of the object (its weight *in vacuo*) and W′ its apparent weight, that is, the weights on the other pan. Let S be the density of the object, S′ that of the weights, and σ that of the air. Then the volume of the object is $\frac{W}{S}$, and the weight of air displaced by it is $\frac{W.\sigma}{S}$. Similarly, the weight of air displaced by the weights is $\frac{W'.\sigma}{S'}$.

So $$W - \frac{W.\sigma}{S} = W' - \frac{W'.\sigma}{S'},$$

i.e., $$W\left(1 - \frac{\sigma}{S}\right) = W'\left(1 - \frac{\sigma}{S'}\right).$$

Therefore $$W = W'\left(1 + \frac{\sigma}{S} - \frac{\sigma}{S'}\right)$$

Since $\frac{\sigma}{S}$ is a small fraction.

In practice the buoyancy correction is obtained from a book of Tables.

Note on the Balance

(1) *Accuracy.*—A balance is true if the beam comes to rest in the horizontal position when any equal weights are placed on the two pans. For this to be the case the arms must be equal in length and, if the weight of the beam is distributed symmetrically, the scale pans must be of equal weight.

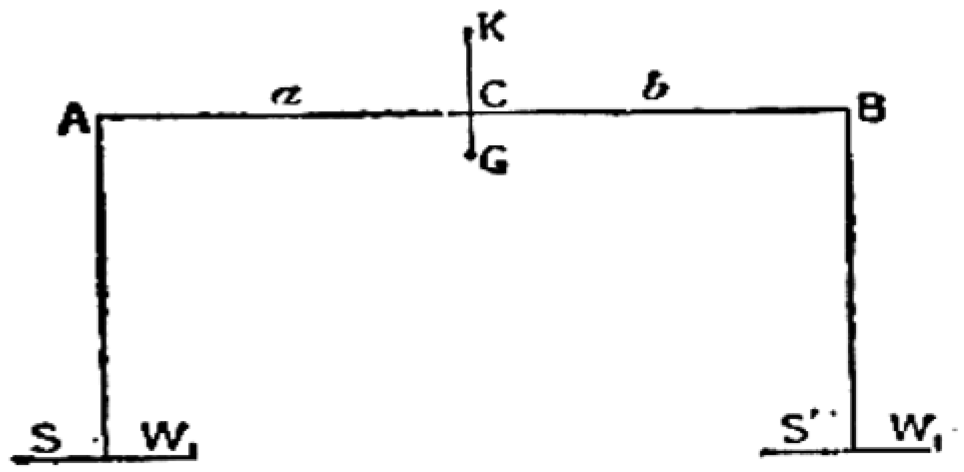

FIG. 39.

To take the general case (Fig. 39) A and B are the scale pan knife-edges and K the central knife-edge. The centre of gravity of the beam, and all parts rigidly attached to it, is at G, KCG being perpendicular to AB. The arms are AC and BC, and the weights of the scale pans S and S′. With equal weights W_1, W_1 on the scale pans the beam is horizontal.

Taking moments about K

$$(W_1 + S)a = (W_1 + S')b.$$

With weights W_2, W_2 we have

$$(W_2 + S)a = (W_2 + S')b.$$

Subtracting $\qquad (W_1 - W_2)a = (W_1 - W_2)b.$

So $a = b$ and from the first equation $S = S'$.

To obtain accurate weighings with an inaccurate balance two methods may be adopted.

(*a*) Weigh the body placed in the left-hand pan and then weigh again in the right-hand pan. If w_1 and w_2 are the weights required to balance in the two cases, the true weight of the body W equals $\sqrt{w_1 . w_2}$. For with Fig. 39 we have

$$(W + S)a = (w_1 + S')b$$

and $\qquad (w_2 + S)a = (W + S')b.$

If $S.a = S'.b$, that is, if the beam is horizontal with no weights on the pans, then $W = \sqrt{w_1 . w_2}$.

(*b*) Place the body on the right-hand pan and on the left place a counterpoise of greater weight than the body to be weighed. Add weights w_1 to the right-hand pan until the beam is horizontal. Remove the body and place weights w_2 on the right-hand pan to balance the counterpoise. The weight of the body is $(w_2 - w_1)$.

(2) *Sensitivity.*—We suppose the balance to be true, that is, the arms have equal length a, and the scale pans equal weight S. Two equal weights W are placed on the pans and on the left pan a small extra weight w. Let the beam come to rest at an angle θ to the horizontal, then the ratio $\frac{\theta}{w}$, or for convenience, since θ is small $\frac{\tan\theta}{w}$, may be taken as a measure of the sensitivity.

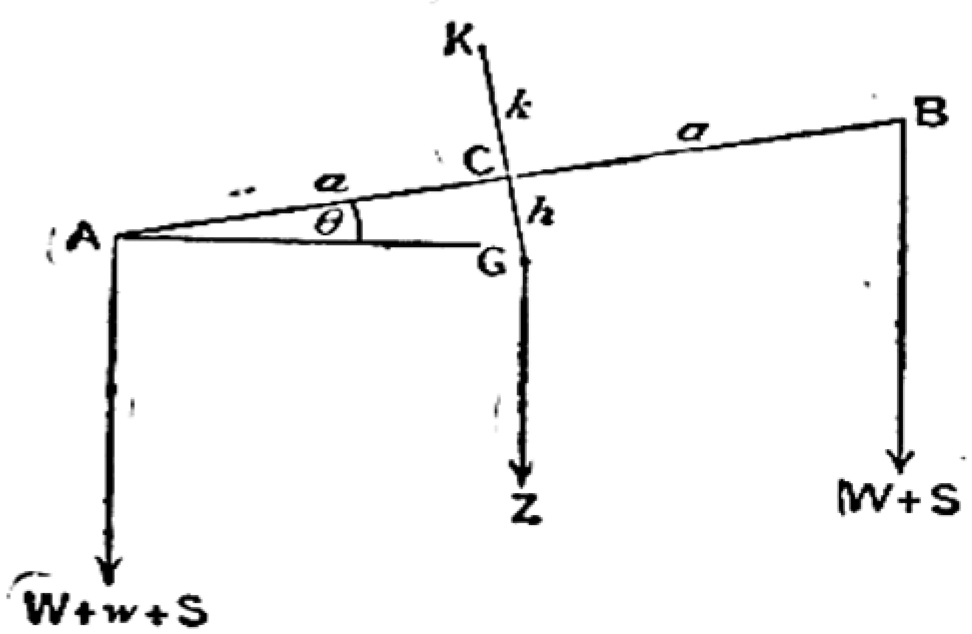

Fig. 40.

Let Z be the weight of the beam. Taking moments about K, the central knife-edge, we have

$$(W+w+S)(a \cos\theta - k \sin\theta) = (W+S)(a \cos\theta + k \sin\theta) + Z(h+k)\sin\theta,$$

or $(W+w+S)(a - k.\tan\theta) = (W+S)(a + k.\tan\theta) + Z(h+k)\tan\theta$.

So
$$\frac{\tan\theta}{w} = \frac{a}{(2W + 2S + w)k + Z(h + k)}.$$

The sensitivity thus increases with a and decreases with increase of W, Z, k and h.

In the balances made for scientific work the three knife-edges are usually in line, so that k is zero. In this case

$$\frac{\tan\theta}{w} = \frac{a}{Z.h}$$

and the sensitivity is independent of the load.

Note that G must be below the central knife-edge for the balance to be stable.

The factors making for high sensitivity in a balance are (1) long arm, (2) light beam, (3) small distance h. Two of these—long arm and small h—have the effect of making the time of swing of the balance large, and so cause the operation of weighing to be slow and tedious.

The weight of the beam cannot be reduced indefinitely, since it must remain rigid.

5. Total Thrust on a Plane Surface in a Liquid

Consider a rectangular lamina ABCD, Fig. 41, immersed in a liquid of density ρ. The edge AB, of length a, is horizontal and at a depth h below the surface. BC, of length b, is inclined at an angle θ to the vertical. The thrust due to the pressure of the liquid on an element of the lamina parallel to AB of thickness δx at a distance x from AB is $a.\delta x.\rho g(h + x \cos \theta)$ absolute units. The total thrust is

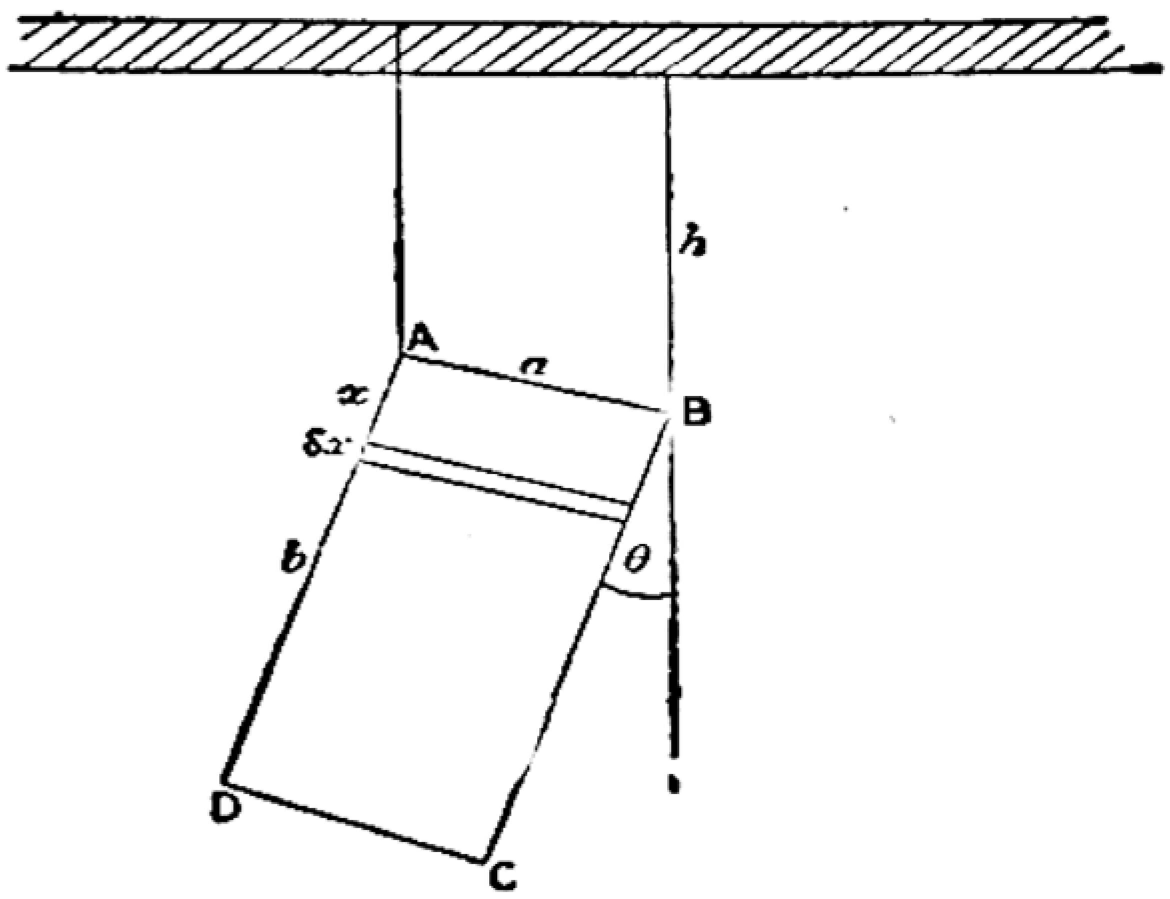

FIG. 41.

$$a\rho g\int_{o}^{b}(h + x \cos \theta)dx = ab\rho g(h + \frac{b}{2} \cos \theta).$$

Note that this is the area of the rectangle times the pressure at its centroid. It is a general rule that the thrust on a plane surface due to liquid pressure is equal to its area multiplied by the pressure at its centroid.

Fig. 42 illustrates the general rule. The plane of the area meets the liquid surface in the line AB, and is inclined at an angle θ to the vertical. The thrust on the typical strip (parallel to the liquid surface) of length y and width δx is $y\delta x.\rho.g.x \cos \theta$, and the total thrust is

$$\Sigma y\delta x.\rho.g.x\cos\theta$$

or $\rho.g.\cos \theta \int xydx$

taken over the area

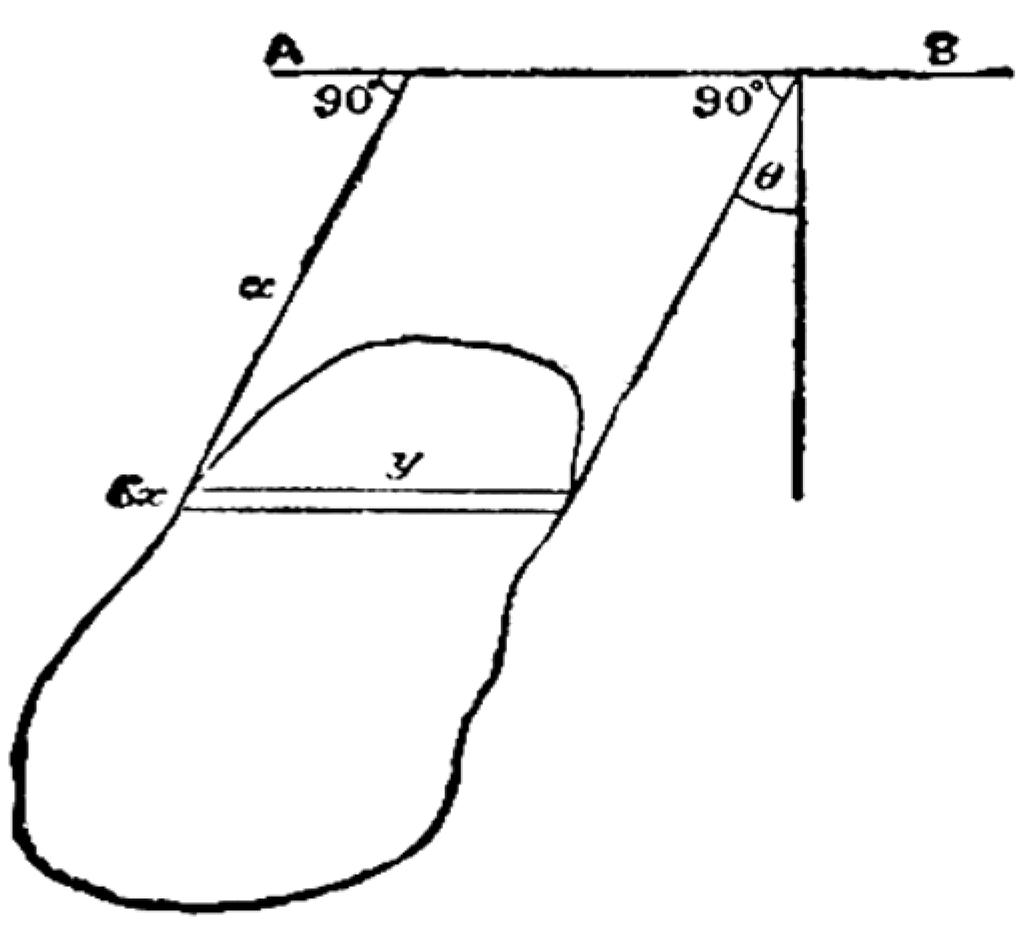

FIG. 42.

Now the area itself is $\int y dx$, and the distance of its centroid from AB is

$$\frac{\int xydx}{\int ydx}.$$

The pressure at the centroid is, therefore,

$$\rho . g . \frac{\int xydx}{\int ydx} . \cos\theta.$$

Multiplying by the area, $\int ydx$, we have $\rho . g . \cos\theta \int xydx$, which is the total thrust on the surface.

To obtain the total thrust on the simpler geometrical areas, either the rule may be applied directly, or the calculation may be performed as above by dividing the area into thin strips parallel to the liquid surface and summing the thrusts on the strips over the whole area by the method of the Integral Calculus.

6. Centre of Pressure

The forces due to liquid pressure on the various elements of a plane surface form a system of parallel forces and have a resultant. The point in which this resultant meets the plane surface is the Centre of Pressure.

A single force, equal to the total thrust on the area, but opposite in direction, would, if applied at the Centre of Pressure, suffice to balance the thrust due to the liquid.

To determine the position of the centre of pressure for the rectangle of Fig. 41 we notice that it must lie on the line parallel to AD which bisects the rectangle.

Let $\bar{x}$ be its distance from AB.

Then $$\bar{x} = \frac{\int_0^b a . \rho . g(h + x\cos\theta) . x . dx}{\int_0^b a . \rho . g(h + x\cos\theta) dx}.$$

(The problem is exactly the same as finding Centres of Gravity.)

$$= \frac{\frac{h . b}{2} + \frac{b^2}{3}\cos\theta}{h + \frac{b}{2}\cos\theta}.$$

If $h = 0$ and $\cos\theta = 1$

then $$\bar{x} = \frac{2}{3} . b.$$

Also, as h becomes very large $\bar{x}$ approaches $\frac{b}{2}$.

It should be noticed that the position of the centre of pressure varies with the depth of immersion.

7. Stability of Floating Bodies. The Metacentre

For vertical displacements the equilibrium of a floating body is stable : since, in order to produce such a displacement, either upwards or downwards, work must be done by some external force, and this involves an increase in the potential energy of the body and liquid. For horizontal displacements the equilibrium is neutral.

For angular displacements the equilibrium may be either stable or unstable and in some special cases it is neutral.

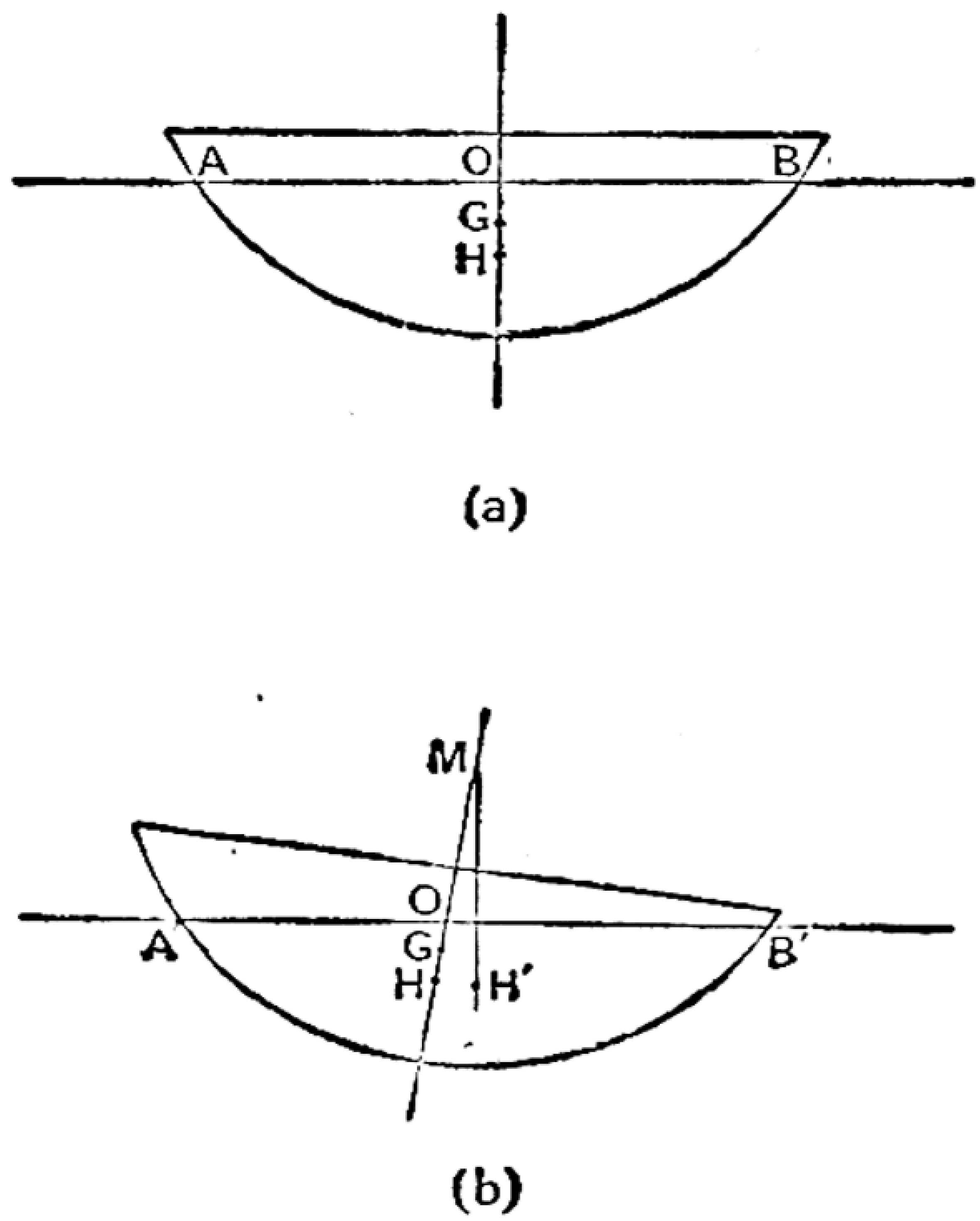

Fig. 43.

Fig. 43 (a) represents a floating body, a boat for example, in its equilibrium position. AOB is the water-line and represents the plane of flotation. G is the centre of gravity of the body and H that of the displaced water. The line GH will be vertical, otherwise the body would not be in equilibrium. Fig. 43 (b) represents the body tilted through a small angle θ. A′OB′ is the new water-line and H′ the new centre of gravity of the displaced water.

Let the vertical through H′ meet HG in M. The forces on the body are now its weight W vertically downwards through G and the force of buoyancy (equal to W) along H′M. These constitute a couple of moment W.GM.sin θ. If M is above G, this couple tends to turn the body back again to the original position, and the equilibrium is stable. If M is below G, the equilibrium is unstable. When θ is very small, M is the metacentre and MG is called the metacentric height. This is a measure of the stability for this type of displacement.

If the displacement is such that there is no change in the volume of liquid displaced, and the body is symmetrical about the vertical plane represented by the plane of the paper or a perpendicular vertical plane, then it can be shown that MH = $\frac{A.k^2}{V}$, where A is the area of the section of the body by the plane of flotation, k is the radius of gyration of this area about an axis perpendicular to the paper through the mean centre of the area, and V is the volume immersed.

For a body totally immersed, such as a submarine, H′ coincides with H and so M also coincides with H. In this case it is necessary for stability only that G should be below H.

Ballast, of course, aids stability by keeping G low.

In practice the metacentric height MG is measured by moving a known weight through a measured distance across the deck and finding the angle through which the ship is turned by means of a plumb-line attached to the mast.

Example.—Examine the stability of a rectangular block of square section, side a, height b, and density ρ, floating in water as shown in Fig. 44.

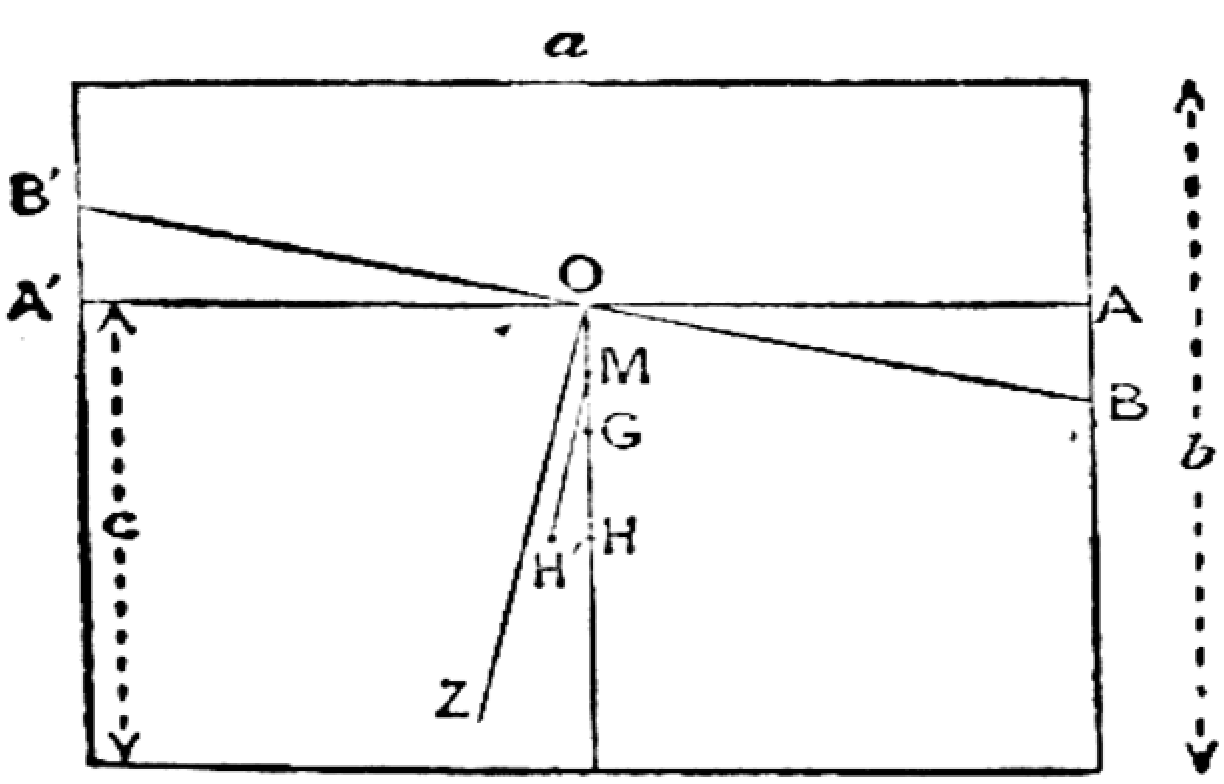

FIG. 44.

AOA′ represents the plane of flotation in the equilibrium position and BOB′ represents this plane after the body has undergone a small angular displacement such that the volume immersed remains unchanged. The body has turned about an axis perpendicular to the paper through the point O such that the triangles AOB and A′OB′ are equal. OZ is the new vertical through O, OGH the old vertical, and ZOH = AOB = θ the small angular displacement. Let H′ be the new position of the centre of gravity of the displaced water. The coordinates $(\bar{x}, \bar{y})$ of H′ with respect to axes OA, HO, are

$$\bar{x} = \frac{a^2c.0 - \frac{1}{8}a^3\theta.\frac{1}{3}a + \frac{1}{8}a^3\theta\left(-\frac{1}{3}a\right)}{a^2.c}$$

$$= -\frac{1}{12}.\frac{a^2\theta}{c},$$

and

$$\bar{y} = \frac{a^2c\left(-\frac{c}{2}\right) - \frac{1}{8}a^3\theta\left(-\frac{1}{6}a\theta\right) + \frac{1}{8}a^3\theta.\frac{1}{6}a\theta}{a^2c}$$

$$= -\frac{c}{2} + \frac{1}{24}\frac{a^2\theta^2}{c}$$

$$= -\frac{c}{2}, \text{ neglecting the term containing } \theta^2.$$

Thus HH′ is parallel to AOA′ and is equal to $\bar{x}$.

Also $$\text{HH}' = \text{MH}.\theta,$$

or $$\text{MH} = \frac{1}{12}.\frac{a^2}{c}.$$

(This could also be obtained immediately from $\text{MH} = \frac{\text{A}.k^2}{\text{V}}$.)

The equilibrium is stable if MH > GH, that is, if

$$\frac{1}{12}.\frac{a^2}{c} > \frac{b}{2} - \frac{c}{2}$$

or $$a^2 > 6b^2.\rho(1 - \rho), \text{ since } \frac{c}{b} = \rho.$$

The greatest value possible for $\rho(1 - \rho)$ is $\frac{1}{4}$, so the equilibrium is stable for all densities of the block between 0 and 1 if

$$a^2 > \frac{6}{4}b^2,$$

that is, if $b < \cdot 82a$.

8. Pressure in a Liquid moving with Constant Acceleration

All particles of the liquid are assumed to be moving with the same acceleration, so there is no relative motion of the parts. The effect of this motion on the hydrostatic pressure will be clear from the following examples.

Example.—A tank of liquid of density ρ gm. per c.c. is carried in a lift moving with an upwards acceleration a cm. per sec. per sec. Find the pressure at a depth h cm. below the surface of the liquid.

Consider a cylinder of the liquid of height h and cross-sectional area 1 sq. cm. having its upper end in the surface and its axis vertical. The forces on it are, its weight $w = 1.h.\rho.g$ dynes, the force due to the atmospheric pressure A, and the upward force p arising from the fluid pressure on its lower end. There are also horizontal forces due to the pressure on the curved sides of the cylinder, but these have no effect on the vertical motion. Now this cylinder of liquid has an upwards acceleration a, so

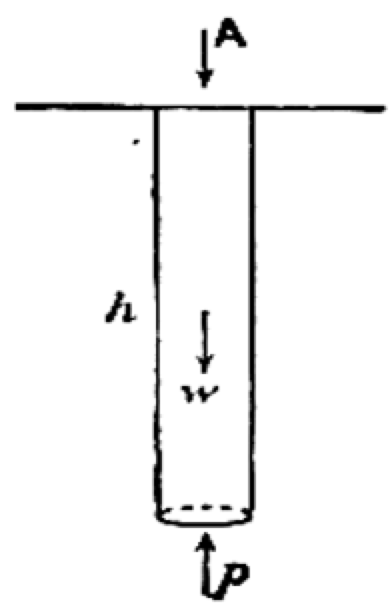

FIG. 45.

$$p - \mathrm{A} - h\rho g = h\rho . a$$

or
$$p = \mathrm{A} + h\rho(g + a).$$

If the acceleration is downwards

$$p = \mathrm{A} + h\rho(g - a),$$

and if the liquid is falling freely, so that $a = g$, the hydrostatic pressure vanishes and the only pressure in the liquid is the uniform atmospheric pressure A.

Example.—Again suppose the tank of liquid has an acceleration a in a horizontal direction. The surface of the liquid will remain plane, but will be tilted at some angle θ to the horizontal. Find θ.

Consider a thin cylinder of length l and cross-section δA, whose axis is horizontal and in the direction of motion. The only forces affecting its motion in a horizontal direction are the forces due to the pressures p_1, p_2 on its flat ends.

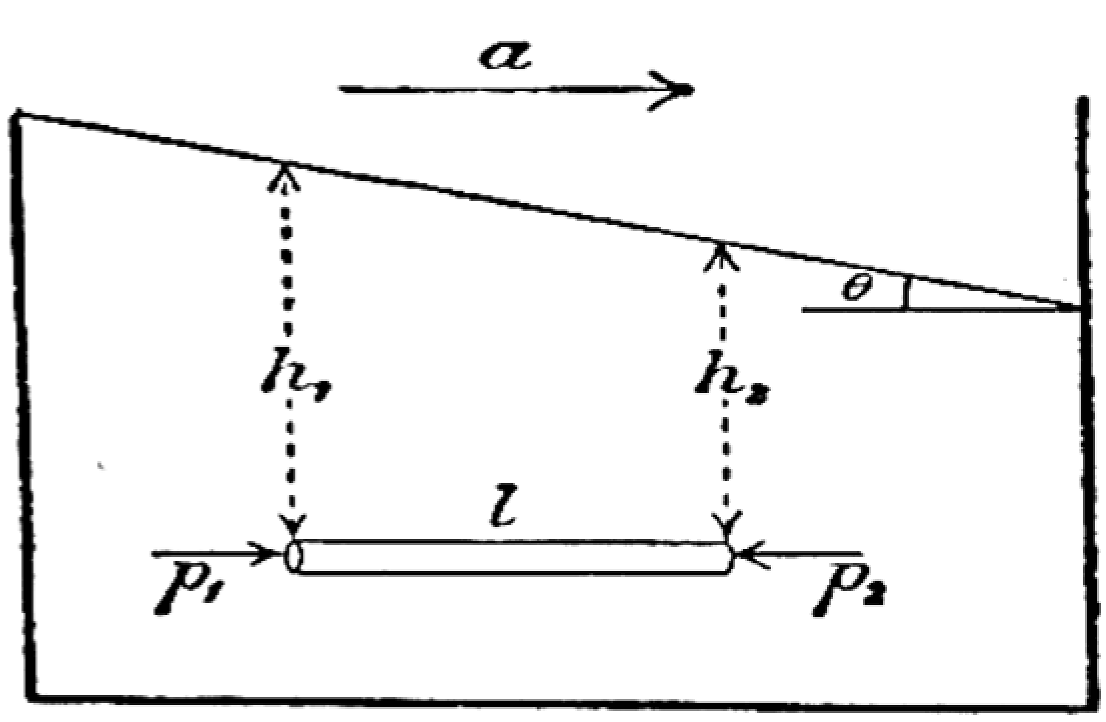

FIG. 46.

We have, then, $(p_1 - p_2)\delta A = l.\delta A.\rho.a.$

But $p_1 = h_1\rho g$ and $p_2 = h_2\rho g.$

so $g(h_1 - h_2) = l.a,$

and $$\tan\theta = \frac{h_1 - h_2}{l} = \frac{a}{g}.$$

Examples

1. Define pressure at a point in a fluid. Find the total thrust on the sides and vertical ends of a V trough 1 ft. deep, 2 ft. wide at the top, and 4 ft. long when filled with water, density 62·5 lb. per c. ft. (O. and C.)

2. Explain what is meant by pressure at a point in a liquid and by centre of pressure. Find the centre of pressure of a square area of side 3 ft. immersed vertically in water with one edge in the surface. Find also the total force on one side of the area, given that the atmospheric pressure is 14 lb. wt. per sq. in., and that a c. ft. of water weighs 62·5 lb. (O. and C.)

3. Find the pressure at a depth x in a liquid of density ρ. Find also if the atmospheric pressure is 14 lb. wt. per sq. in. and if a c. ft. of sea-water weighs 62·5 lb., the depth in the sea at which a pressure of 20 atmospheres is reached. (O. and C.)

4. Prove that the pressure at a point in a liquid is the same in all directions. Find the resultant pressure and the centre of pressure for each of the sides of a cube of which each side is a square metre, and which is filled with water. (O. and C.)

5. State and prove the principle of Archimedes. A body floats in a liquid in a closed vessel. Discuss the effect of changing the pressure of the air in the vessel. (O. and C.)

6. A piece of an alloy of copper and tin weighs 260 gm. in water and 240 gm. in a liquid of sp. gr. 1·5. Assuming that the volume of the alloy is the sum of the volumes of its constituents, find the masses of copper and tin in the given piece of alloy. Sp. gr. of copper = 8·9, of tin = 7·3. (N.U.)

7. A constant weight hydrometer has a mass of 20 gm. and reads 1·00 when immersed in water. A piece of brass (sp. gr. 8·4) is attached to the lower end by a brass wire, so that the total mass is increased to 22·5 gm., and the level to which the hydrometer sinks in water is noted. What is the sp. gr. of the liquid in which, without the brass, the hydrometer would sink to the same level? (N.U.)

8. State and give a theoretical proof of Archimedes' Principle.

A glass test-tube of mass 10 gm. and sp. gr. 2 is immersed open end downwards in water, so that it contains 6 c.c. of air. If the atmospheric pressure is 76 cm. of mercury, find the increase of pressure required before the tube sinks.

9. Describe the construction of an accurate balance and carefully state upon what factors the sensitiveness depends. A hollow air-tight 2-litre sphere was carefully counterpoised on a balance by lead shot and left undisturbed for a day or two. On returning to the balance it was observed that the sphere was no longer counterpoised and an additional weight of 0·034 gm. had to be added to the lead shot to restore the balance. The temperature on each occasion was exactly 0° C. Offer some explanation. What additional data are necessary for a complete explanation? Assume such data and deduce the conclusion. (N.U.)

10. The distance between the scale pan knife-edges in a balance is 40 cm. The central knife-edge is at a perpendicular distance of 1 cm. above the middle point of the line joining the scale pan knife-edges, and the centre of gravity of the beam is 1 cm. below the middle point of the same line. Find the deflection of the beam when weights of 10·0 and 10·1 grams are placed in the scale pans. The weight of the beam is 1,000 grams and the scale pans weigh 25 grams each. Would the deflection be the same if the weights in the scale pans were 1·0 and 1·1 grams? Give your reasons. (N.U.)

11. Discuss the problem of the stability of partially immersed bodies floating in a liquid.

In most ships the centre of gravity of the ship lies above the centre of buoyancy. Explain the stability of these ships with respect to rotation, that is, rolling motion. (O. Schol.)

12. A cylinder of radius r and height h, whose density is ρ, floats with its axis vertical in a liquid of density σ. Show that the equilibrium is stable if $r^2 > 2h^2\left(\frac{\rho}{\sigma} - \frac{\rho^2}{\sigma^2}\right)$.

13. The barometer reads 76 cm. of mercury (density 13·6 gm. per c.c.). The density of a liquid is 1·1 gm. per c.c., and the depth of a point A in this liquid is 30 cm. What vertical acceleration of the liquid will reduce the fluid pressure at A by 1% of the value when the liquid is at rest? (N.U.)

CHAPTER VI

SURFACE TENSION

1. Some Effects of Surface Tension

THERE are a number of common phenomena which appear not to agree with the simple laws and principles of Hydrostatics as discussed in the last chapter.

For example, a steel razor-blade will float on water if dropped so as to fall flat on the surface.

Again, if a capillary tube, open at both ends, is held with one end dipping into water, the level of the water in the tube is considerably higher than the level outside.

Near the boundary of a containing vessel the surface of a liquid has a sharp curvature.

Small quantities of mercury, water, etc., resting on a flat surface, gather up into globules instead of spreading over the surface.

It is possible to form a film of liquid (water or soap solution) on a rectangular frame of wire by simply dipping the frame into the liquid and withdrawing it. If a loose piece of cotton AB is tied to the frame as in Fig. 47, it will float loosely on the film. When the film on the right of the cotton is broken, the cotton becomes taut and takes the form of an arc of a circle. If a loop of cotton is floated on the film, on breaking the film inside the loop the cotton is pulled into the shape of a circle. This suggests that the film is in a state of tension and exerts an inwards force on its boundary at right-angles to the boundary.

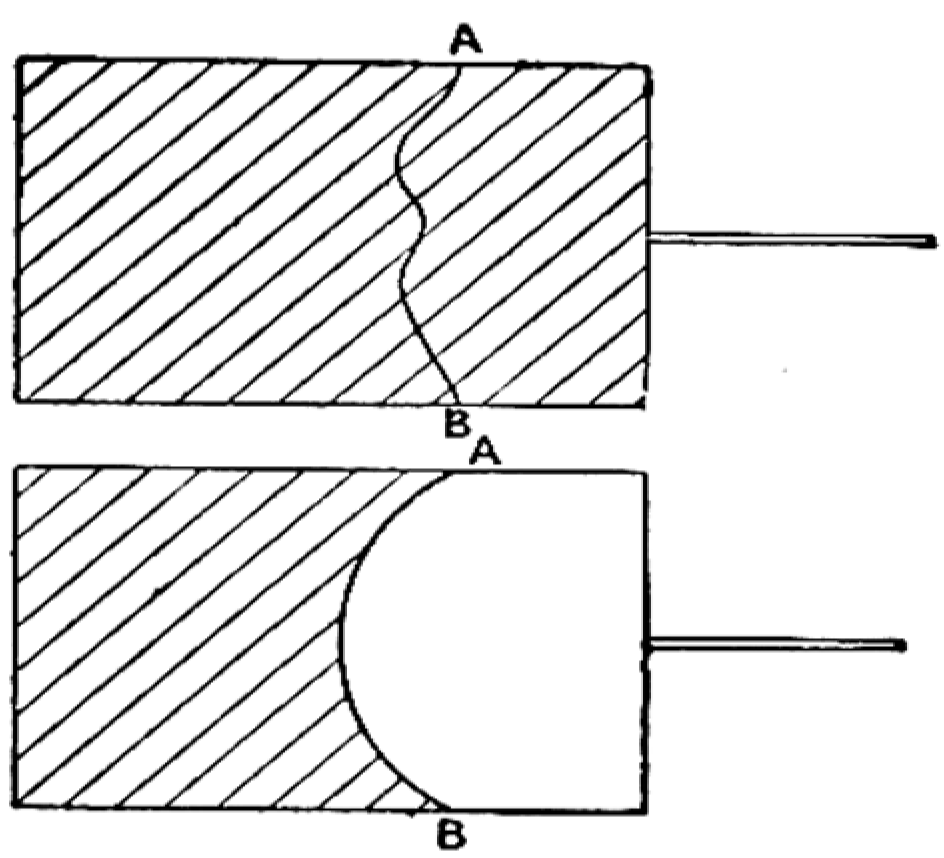

FIG. 47.

This state of tension in the surface of a liquid is also suggested by the following experiment. A quantity of aniline (a few c.c.)

may be introduced by means of a pipette into a solution of common salt in water, of such strength that its density is the same as that of aniline. The aniline does not mix with the solution, and, after a few oscillations, will take the form of a sphere, floating in the body of the solution. The spherical shape is, of course, that which has the least surface for a given volume.

The above phenomena are exactly what would be found if the surface of a liquid, or the surface of separation of two immiscible liquids, were a thin membrane in a state of constant tension, equal in all directions. Such a membrane, clearly, would have a tendency to contract its area.

This idea lends itself to calculation, and the results of calculation agree with those of experiment. The successive shapes of a slowly forming drop, up to the moment of breaking away, also suggest the idea of a membrane. There is an important difference, however, between the behaviour of an elastic membrane and that of the surface layer of a liquid. The tension of an elastic membrane increases with stretching; the tension of the surface layer of a liquid remains constant (so long as its temperature is constant) whilst its area is changed.

2. Laplace's Theory of Capillarity or Surface Tension

Laplace (1749–1827) explained this tension in the surface of a liquid by the supposition that the molecules of a liquid exert a force of attraction on each other (in addition to and independent of their gravitational attraction), which, though large when the molecules are close together, becomes negligible beyond a certain small distance, C, called the range of molecular action. If a sphere of radius C be described with some particular molecule in the body of the liquid as centre, this molecule is attracted by all molecules inside the sphere, but is entirely uninfluenced by those outside the sphere. The resultant force, due to these attractions, on a molecule in the body of the liquid is zero, since it is pulled equally in all directions. But for a molecule, closer to the surface than C, there will be a resultant force towards the body of the liquid. In Fig. 48 this force on a molecule P is

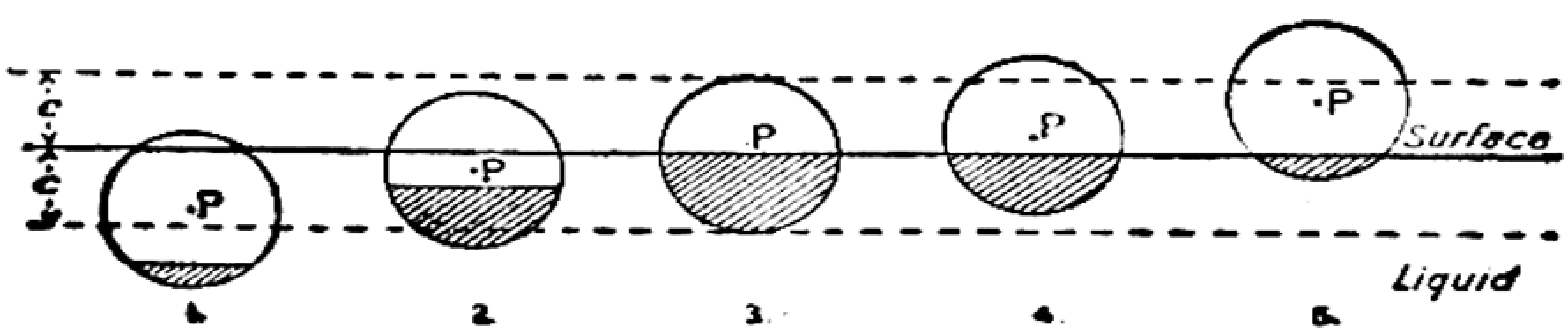

FIG. 48.

that due to the attraction of molecules inside the shaded part of the sphere, radius C and centre P, the remainder of the liquid inside the sphere having no resultant force on P. It should be noticed that the attraction is exerted also on a molecule of vapour above the liquid surface, so long as its distance from the surface is less than C, and that the attractions in cases 1 and 5 are equal, and also in 2 and 4.

The surface layer of the liquid of thickness C is thus in a state of stress which is manifest as a tension in the surface, much in the same way as a heavy string, suspended from two points so as to be nearly horizontal, is in a state of tension as a result of the action of gravity.

This theory of Laplace also indicates relations between the surface tension of a liquid, its latent heat of vaporisation, and its intrinsic or cohesion pressure. Some account of these relations is given in a later section of this chapter.

3. Definition of Surface Tension

Let a straight line XY be imagined on the surface of a liquid, dividing it into two regions A and B. The surface layer of A will exert a force at right angles to XY on the portion B. B will exert an equal and opposite force on A. The magnitude of this force in dynes per cm. length of XY is the surface tension of the liquid.

Suppose we have a rectangular frame of wire, the side AB being able to slide without friction along the adjoining sides. If the frame supports a liquid film, a force F dynes (Fig. 49) will have to be applied to keep AB in position. For equilibrium we have $F = 2.T.AB$, since the film has two surfaces, each exerting a pull on AB. T in this equation is the surface tension of the liquid. It is the force on AB per cm. length due to one surface.

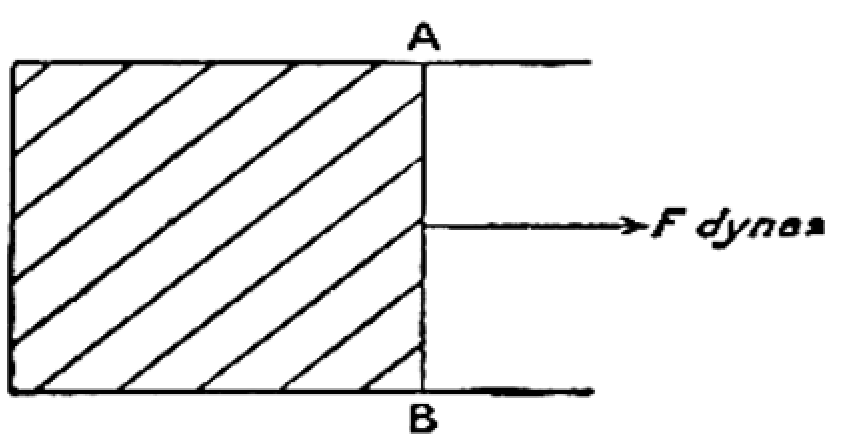

FIG. 49.

Surface tension always acts at right angles to the boundary of the surface and is the same in all directions in the surface.

Work Done in Stretching a Film at Constant Temperature. Definition of Surface Energy

In Fig. 49, if we imagine the force F to move AB through a small distance δx in its own direction, the work done will be

$F.\delta x = 2T.AB.\delta x = T.\delta a$, where δa, which stands for $2AB.\delta x$, is the new surface created.

There is an increase in the potential energy of the film of amount $T.\delta a$, or T ergs per sq. cm. of increase in area.

We have considered here only an infinitesimal increase in area. If the area of the film is increased by a finite amount the film will be cooled unless heat is supplied to it from its surroundings, and similarly if the film contracts its temperature will rise. This is a consequence of the experimental fact that the surface tension of liquids in general decreases with rise of temperature. The cooling of the film, when extended by an external force, is an example of Le Chatelier's Principle, that a system subjected to a change in its state reacts in such a way as to oppose or nullify the change. Stretching the film causes it to fall in temperature, and so increases its surface tension and produces a greater resistance to the extending force.

If we consider the finite extension of the film to be carried out slowly, sufficient heat will be received from the surroundings to keep the temperature constant. In this case the surface tension T will remain constant, and the work done, as AB moves over the distance x, will be

$$F.x = 2T.AB.x = T.a,$$

where a is the newly-formed area of liquid surface.

This work done is equal to the increase in Potential Energy. That is, $T.a = S.a$, where S denotes the Potential Energy in ergs per sq. cm.

So numerically $T = S$.

(Note also that they have the same dimensions in mass, length and time.)

This quantity S, which is the potential energy of the surface per unit area under the condition of constant temperature, is usually called the 'surface energy.'

A consideration of the heat changes, which accompany changes in area of the film, will throw further light on the meaning of 'surface energy.' Suppose the force F in Fig. 49 to be decreased by an infinitesimal amount. The surface tension will pull AB to the left and the film will contract to zero area, or, at any rate, to a very small area. The work done by the film in this contraction depends on the conditions under which it is carried out. If we suppose the film to be thermally insulated, so that no transference of heat takes place between it and its surroundings, then, in contracting the film will rise in temperature, the

surface tension will diminish, and the work done by the film will be less than would have been done if the surface tension had maintained its initial value. The initial energy of the film has resulted in a certain amount of work done and in some heat developed in the film.

If the contraction takes place in such a manner that the temperature of the film remains constant (*i.e.*, the film can freely give up any heat developed to its surroundings), then, in this case the initial energy of the film results in a definite amount of work and some heat which has passed to the surroundings. The heat which passes to the surroundings in the isothermal contraction is less in amount than that developed in the film in the adiabatic contraction, and the work done by the film is greater in the isothermal case.

The student must distinguish between the 'surface energy,' S, and what has been called in this section the initial energy of the film (reckoned per sq. cm.). The 'surface energy' may be defined as the work done by the film in an isothermal contraction to zero area divided by its area. This is a smaller quantity than the initial energy of the film per unit area, because part of the initial energy has been converted into heat.

With this meaning the 'surface energy,' S, is equal to T, the surface tension.

It is in this sense that the term is used in the remainder of the chapter.

4. Surface Tension and Temperature

The surface tension of a liquid can be measured by various methods to be described later. It is found that for all liquids the surface tension decreases with rise of temperature. We should expect it to become zero at the critical temperature. At temperature $t°$, the surface tension, T, is given approximately by the formula

$$T = K(t_c - t),$$

where t_c is the critical temperature and K is a constant for the liquid.

5. Excess Pressure Inside a Bubble and Drop

Let us calculate the pressure in a soap-bubble in terms of the surface tension T and the radius r. Fig. 50 shows the forces acting on one half of the bubble. P is the pressure inside · A,

the atmospheric or external pressure. The resultant of the pressures is

$$(P - A)\pi r^2 \text{ upwards.}$$

This is balanced by the force due to surface tension round the rim of the hemisphere, viz., $2T.2\pi r$, the bubble having two surfaces. The weight of the film is neglected as being extremely small.

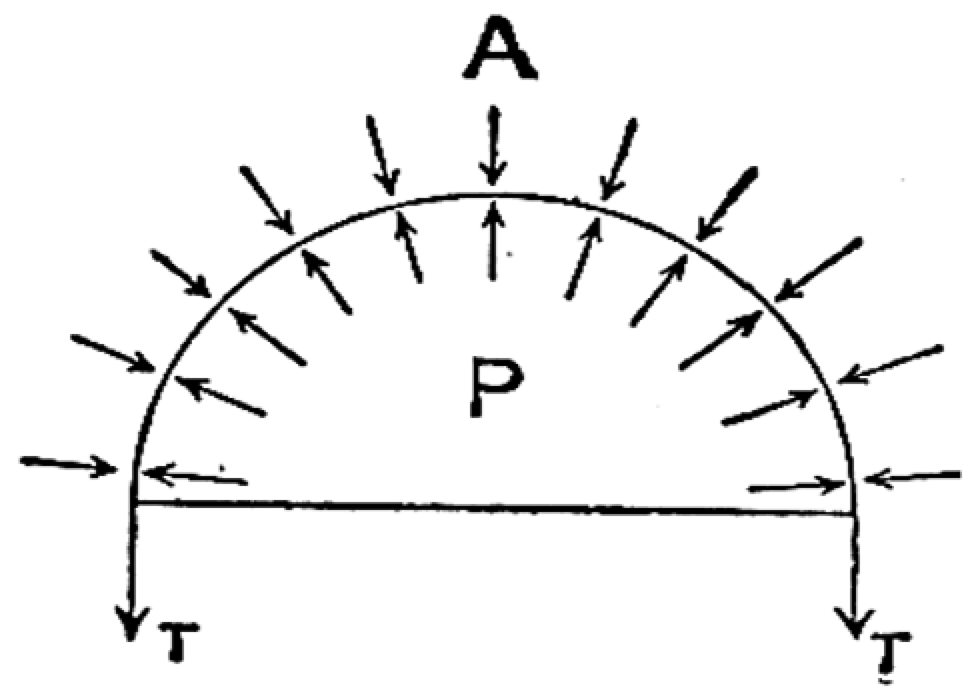

FIG. 50.

Hence $$(P - A) = \frac{4T}{r}.$$

In the case of a spherical drop with only a single surface we have

$$(P - A)\pi r^2 = T.2\pi r$$

$$P - A = \frac{2T}{r}.$$

The pressure P in this case acts over the circular face of the hemisphere, and the drop may be considered to be falling freely.

The expressions for the difference of pressure between the two sides of a spherical film or liquid surface also hold when the surface is only a part of a sphere, a spherical cap.

Example 1.—A film forming part of a sphere of radius r cm. is formed on the end of a tube of radius d cm., Fig. 51. To maintain the film in equilibrium the pressure on its inside surface must be greater than that outside.

The film pulls the edge of the tube with a force T dynes per cm. length of the circumference of the tube at right-angles to the radius of the film. The tube exerts equal and opposite forces on the film, represented by TT in the figure. Resolving in the vertical direction, we have

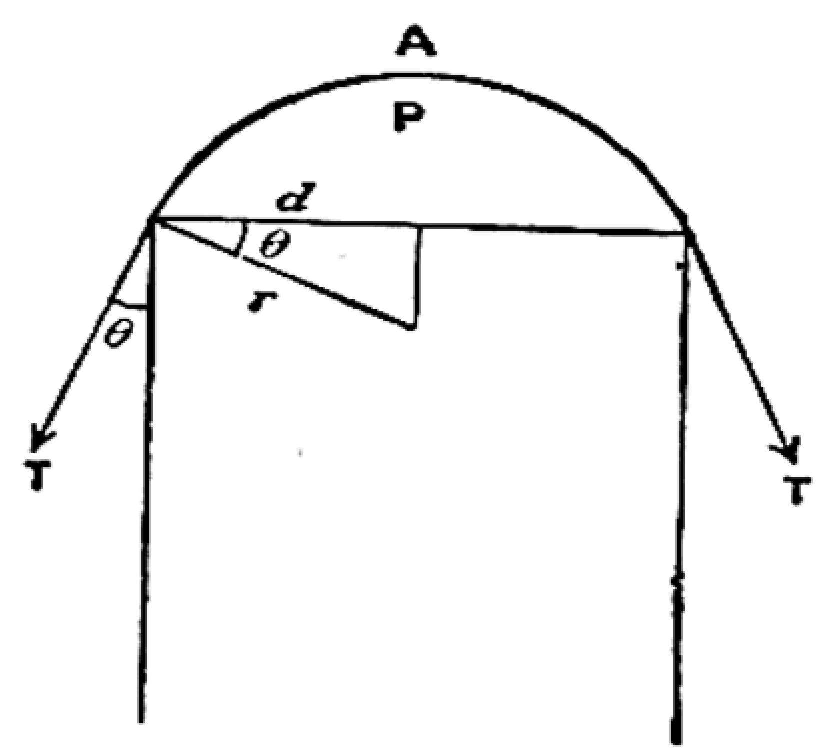

FIG. 51.

$$(P - A).\pi.d^2 = 2T.2\pi d.\cos\theta$$

$$= 2T.2\pi d.\frac{d}{r}$$

or $$P - A = \frac{4T}{r}.$$

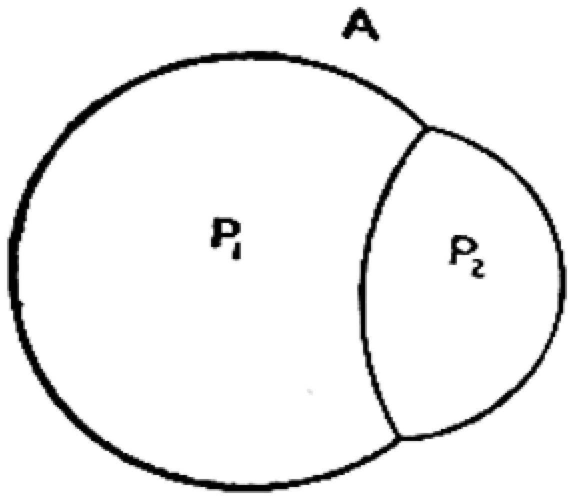

Fig. 52.

Example 2.—Two soap-bubbles of radii 3 cm. and 2 cm. have joined together so as to have a portion of their surfaces in common. Find the radius of this common surface.

Let the pressures be as shown in Fig. 52, and let r be the radius of the common portion. We have

$$P_1 - A = \frac{4T}{3}$$

$$P_2 - A = \frac{4T}{2}$$

and $$P_2 - P_1 = \frac{4T}{r}.$$

Whence $$\frac{1}{r} = \frac{1}{2} - \frac{1}{3} \text{ and } r = 6 \text{ cm.}$$

The pressure in the small bubble will be greater than that in the larger bubble, and the shape of the common surface will be as shown in the diagram.

The Principle of Virtual Work

The calculation of the excess pressure inside a soap-bubble may be performed with the aid of some form of the Principle of Virtual Work.

The Principle states that :—

If a system in equilibrium be imagined to undergo any infinitely small displacement, consistent with the geometrical conditions of the system, the work done by the forces acting on the system will be zero.

This means that a small displacement of the system from a position of equilibrium produces no change in the potential energy.

The potential energy, then, of a system in equilibrium has a stationary value. If the potential energy is a maximum, the equilibrium is unstable ; if the potential energy is a minimum, the equilibrium is stable. When the potential energy, being stationary, is neither a maximum nor a minimum, the equilibrium is neutral.

The term, potential energy, in the above, means the total potential energy of the system arising from gravity, surface tension, state of strain and electrical condition.

For example, the shape assumed by a small drop of mercury is such that the total potential energy is a minimum. The potential energy due to gravity would be decreased if the drop were to spread; but this would involve an increase in area of surface and therefore in the potential energy due to surface tension. The actual shape makes the sum of these potential energies a minimum.

If we confine our attention to one form of potential energy in the system, the work done in a small displacement from an equilibrium position by the remainder of the forces, which do not contribute to this potential energy, will be equal to the increase in the potential energy.

To illustrate this let a soap-bubble of radius r undergo a displacement which increases its radius to $r + \delta r$. Let P and A be the internal and external pressures, and let S be the surface energy per unit area, and T the surface tension.

The increase in area of the film, which has two surfaces, will be

$$2\delta(4\pi r^2) = 2 . 8\pi r . \delta r.$$

The increase in energy $= 16\pi r . \delta r . \mathrm{S}.$

If $\delta\mathrm{V}$ is the increase in volume, the work done by the pressures

$$= (\mathrm{P} - \mathrm{A}) . \delta\mathrm{V}$$

$$= (\mathrm{P} - \mathrm{A})\delta\left(\frac{4}{3}\pi r^3\right) = (\mathrm{P} - \mathrm{A}) . 4\pi r^2 . \delta r.$$

We have, then, $(\mathrm{P} - \mathrm{A}) . 4\pi r^2 . \delta r = 16\pi r . \delta r . \mathrm{S}.$

That is, $$\mathrm{P} - \mathrm{A} = \frac{4\mathrm{S}}{r} = \frac{4\mathrm{T}}{r},$$

since $\mathrm{S} = \mathrm{T}$.

The first form of the Virtual Work Principle might be used.

When the area of the film is increased, work is done against surface tension. The work done by surface tension is then

$$- 2\mathrm{T} . \delta(4\pi r^2) = - 16\mathrm{T} . \pi r . \delta r.$$

The total work done by the forces is, therefore,

$$(\mathrm{P} - \mathrm{A}) . \delta\mathrm{V} - 16\mathrm{T} . \pi r . \delta r$$

$$= (\mathrm{P} - \mathrm{A})4\pi r^2 . \delta r - 16\mathrm{T} . \pi r . \delta r = 0.$$

Therefore $$\mathrm{P} - \mathrm{A} = \frac{4\mathrm{T}}{r}.$$

The Virtual Work Principle often provides a neat solution of problems concerned with the equilibrium of a system.

6. Difference of Pressure between the Two Sides of a Film or Liquid Surface

A film may have many other forms than that of a spherical surface. The difference in pressure between the two sides of the film is in general $2T\left(\frac{1}{r}+\frac{1}{r'}\right)$, where r and r' are the two principal radii of curvature of the surface at any point. If the difference in pressure is the same all over the film, then, its shape must be such that, at every point, $\frac{1}{r}+\frac{1}{r'}$ is constant. If the pressure is the same on both sides of the film, it has the property that at all points $\frac{1}{r}+\frac{1}{r'}=0$.

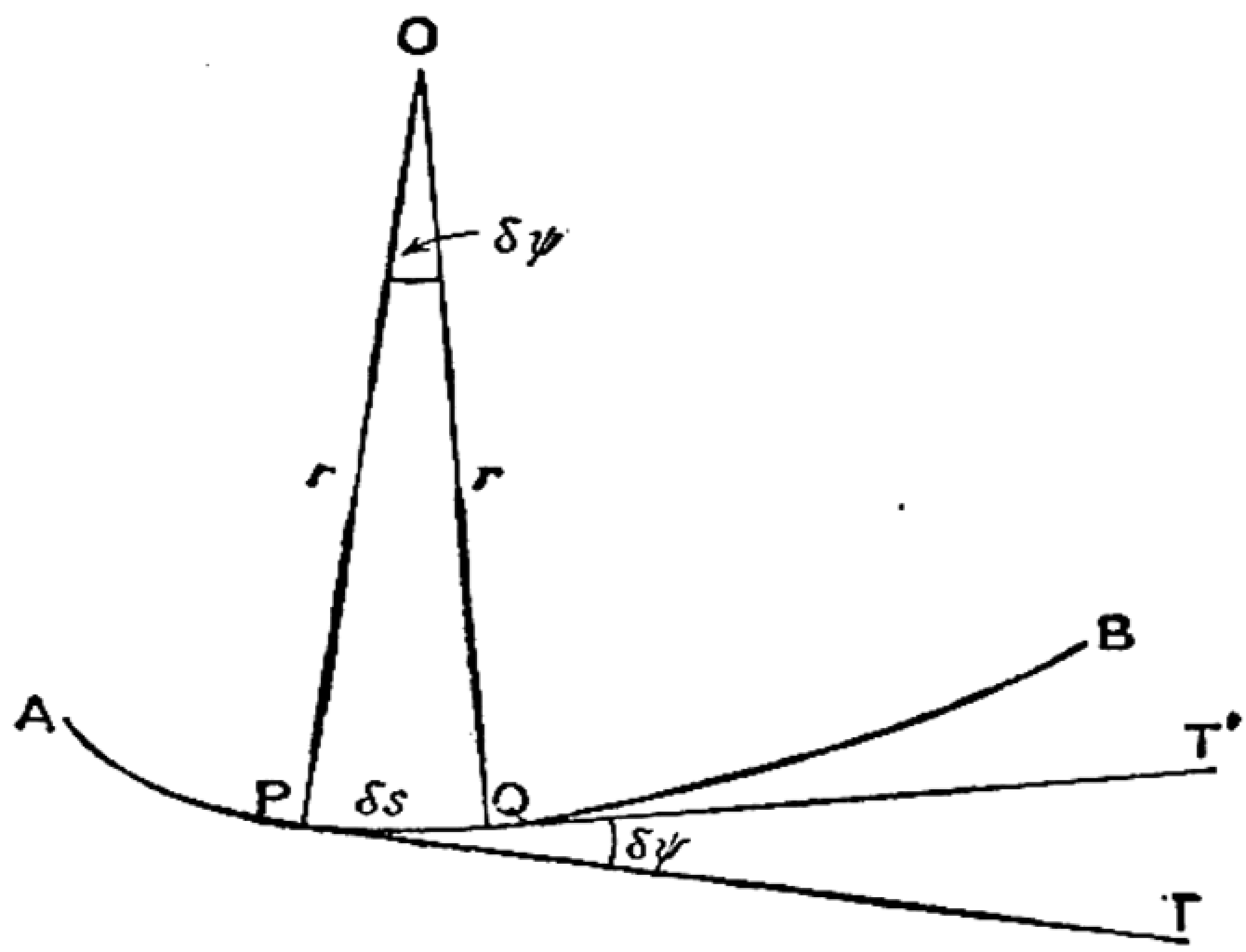

FIG. 53.

To appreciate this result it is necessary to understand the meaning of curvature. In Fig. 53 PQ is an element, length δs, of any plane curve AB. PT, PO are the tangent and normal at P; QT', QO, the tangent and normal at Q. The curvature at P is the change of direction of the curve per unit length of arc near that point, or, the curvature at P

$$=\mathrm{L}t_{\delta s\to 0}\frac{\delta\psi}{\delta s}=\frac{d\psi}{ds}=\frac{1}{r},$$

where r, the radius of curvature, is the length PO when $\delta s\to 0$. The point O is then the centre of curvature for the point P. For a circle the curvature is the same at all points and is the reciprocal of the radius; for a straight line it is, of course, zero.

Consider now the part of the surface near B, Fig. 54, of a Rugby football. OB is the normal to the surface at B, *i.e.*, the perpendicular to the tangent plane at B. A series of planes passing through OB cut the surface near B in a set of curves intersecting at B. These curves have different curvatures at B. It is clear that the portion of the circle CBC′ has the greatest curvature and the portion of the elliptic arc ABA′ has the least curvature at B. It is true generally for any point on a surface that two curves will be found in the above manner, one having maximum curvature, the other, minimum curvature. These two curves are always, as in the case above, at right-angles to each other. Their radii of curvature are called the principal radii of curvature of the surface for that point.

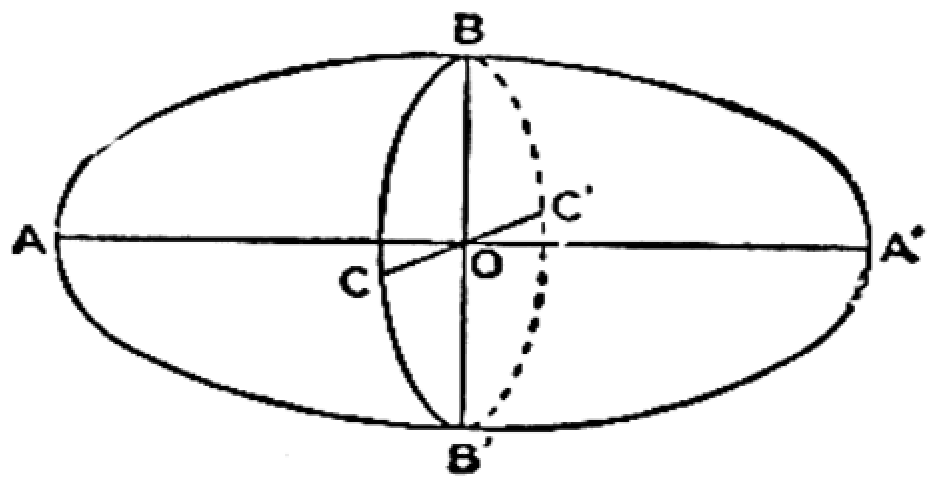

FIG. 54.

In Fig. 55 ABCD represents a small portion of a film, AOO′ being the normal at A. AB, AD are very small arcs of the

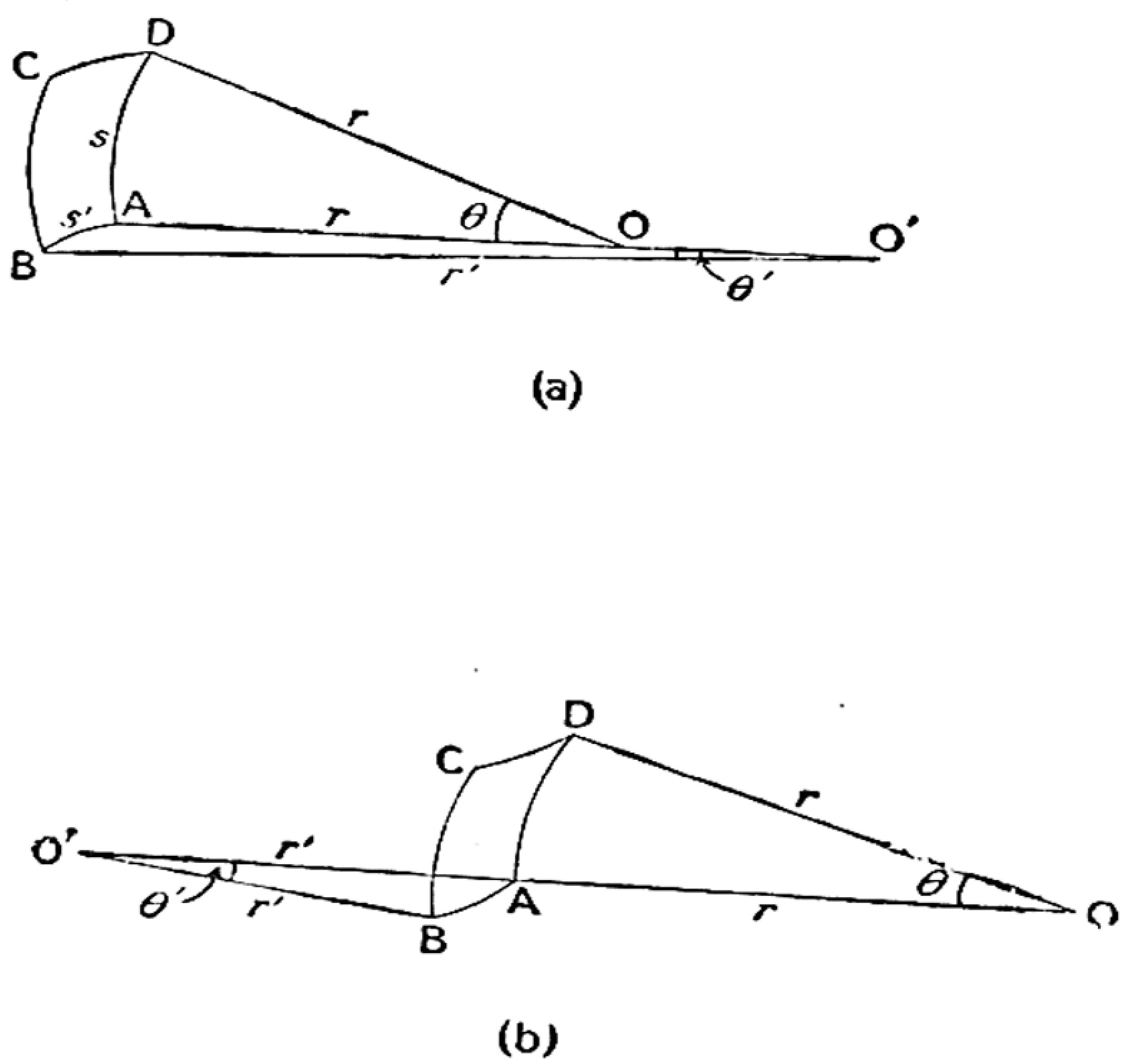

FIG. 55.

curves of principal curvature ; r, r' are the principal radii of curvature. r, r' are finite lengths ; s, s' are small lengths of arc ; θ, θ' are small angles.

Let the area ABCD $= a$, then $a = s.s' = r.\theta.r'\theta'$. Suppose the surface undergoes a small displacement such that each point of it moves a distance δh outwards along the normal at that point. Then θ and θ' remain constant, whilst r and r' increase by δh. The resulting increase in a will be

$$\begin{aligned}\delta a &= \theta.\theta'.\delta(r.r') \\ &= \theta.\theta'\{r.\delta r' + r'.\delta r\} \\ &= \theta.\theta'.\delta h(r + r'),\end{aligned}$$

since

$$\delta r = \delta r' = \delta h.$$

For a film, then, the work done by surface tension will be

$$-2\mathrm{T}.\delta a = -2\mathrm{T}.\theta.\theta'.\delta h(r + r').$$

The work done by the pressures on the two sides of the film will be

$$(\mathrm{P} - \mathrm{A}).a.\delta h = (\mathrm{P} - \mathrm{A})\theta.\theta'.r.r'.\delta h,$$

and we have, since the total work done is zero,

$$(\mathrm{P} - \mathrm{A}).r.r' = 2\mathrm{T}(r + r'),$$

or

$$\mathrm{P} - \mathrm{A} = 2\mathrm{T}\left(\frac{1}{r} + \frac{1}{r'}\right).$$

And for a single liquid surface

$$\mathrm{P} - \mathrm{A} = \mathrm{T}\left(\frac{1}{r} + \frac{1}{r'}\right).$$

The centres of curvature may be on opposite sides of the surface as in Fig. 55 (b). In this case, if r is on the side where the pressure is P, r is positive and r' is negative.

If there is no difference in pressure on the two sides of the film, then, at every point $\frac{1}{r} + \frac{1}{r'} = 0$, and the principal radii of curvature are everywhere equal and opposite.

A soap film formed on a wire frame bent into any irregular shape, not confined to one plane, will have this property. Other films of this type may be formed between the wide ends of two funnels, the narrow ends being open to the atmosphere.

For a cylindrical film of radius r we have $\frac{1}{r'} = 0$, so $(\mathrm{P} - \mathrm{A}) = \frac{2\mathrm{T}}{r}$, and for a single surface $\mathrm{P} - \mathrm{A} = \frac{\mathrm{T}}{r}$.

7. Angle of Contact

If a small quantity of water or benzine is placed on the surface of a clean glass plate, it is found to spread over the whole surface. Ethyl alcohol and methyl alcohol behave in the same way. Mercury, turpentine, paraffin, ether, and many other liquids do not spread, but form pools or drops with definite boundaries, and the liquid surface meets the glass surface at a definite angle. For paraffin this is 26°, for turpentine 17°, for ether 16°, for mercury it is about 140°. For a plate of other material these angles would be different. The same angles are observed where the liquids are in contact with the walls of a capillary tube or with the sides of a glass vessel. They are called the angles of contact of the liquids with glass. The angle of contact for liquids which spread on glass is zero.

Some explanation of these facts can be given if we assume the existence of further surface energies in the boundaries between solid and liquid, and between solid and air. For the boundary between liquid and air the surface energy is numerically equal to the surface tension of the liquid.

Let us denote these surface energies per sq. cm. in the order above by S_1, S_2, and S. In Fig. 56 XX represents the solid surface, YY that of the liquid; θ is the angle of contact. If YY receives a small displacement into the position of the dotted line, through a distance δx along the solid surface, there will be a change in the areas of the surfaces, and consequently in their energies. The total change in the energy will be zero. For 1 cm. width perpendicular to the paper we have, then,

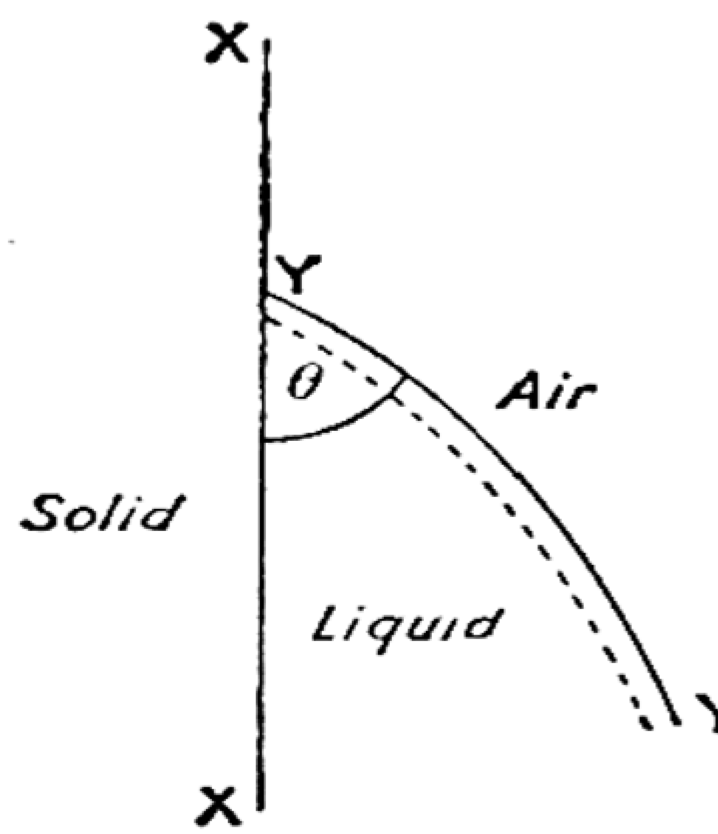

FIG. 56.

$$S_2 . \delta x - S_1 . \delta x - S . \delta x . \cos\theta = 0,$$

or

$$\cos\theta = \frac{S_2 - S_1}{S}.$$

Spreading of the liquid occurs if $\cos\theta = 1$, that is, if

$$S = \text{or} < S_2 - S_1.$$

Little more can be said about S_1 and S_2—except that they certainly exist—since we have, as yet, no method of measuring them.

8. Rise of Liquid in a Capillary Tube

Let the angle of contact be θ. If the radius r of the tube be small compared with h, the rise (see Fig. 57), we may consider the surface of the meniscus to be part of a sphere of radius R, where $r = R \cos \theta$. Let A denote the atmospheric pressure, and

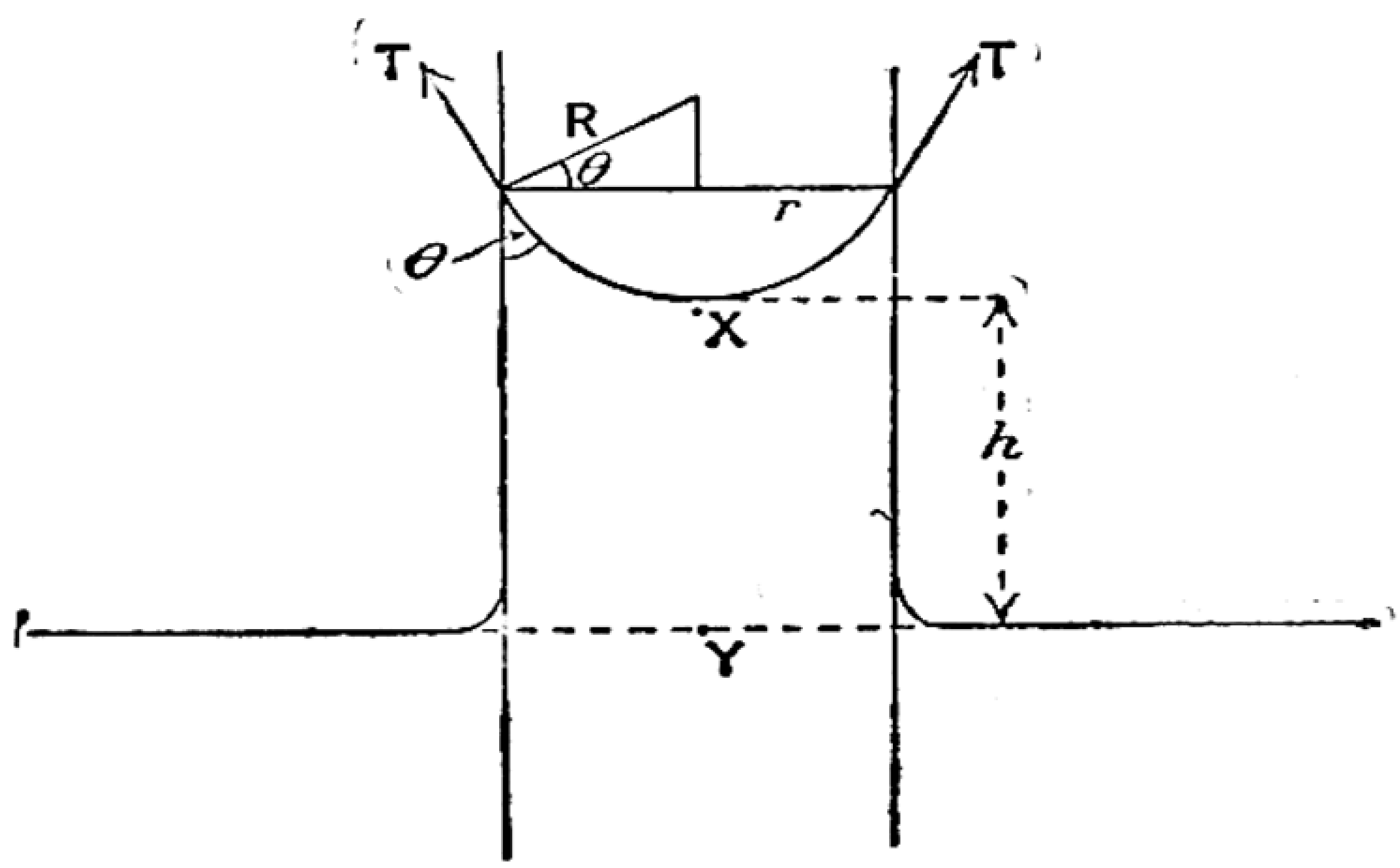

FIG. 57.

P the pressure at a point X just below the meniscus. Then A is greater than P. Also the pressure at Y is A, and we have

$$A = P + h.\rho.g,$$

where ρ is the density of the liquid.

But
$$A - P = \frac{2T}{R} = \frac{2T.\cos \theta}{r}.$$

Therefore
$$h.\rho.g = \frac{2T.\cos \theta}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In the case of mercury, $\cos \theta$ is negative, and h is therefore negative. The meniscus is convex from above and the level of the mercury in the tube is depressed below the level outside.

Equation (1) may also be obtained by resolving vertically the forces on the column of liquid inside the capillary tube.

We have
$$T \cos \theta.2\pi r = \pi r^2.h.\rho.g, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

omitting the force due to atmospheric pressure which acts equally and oppositely on the top and on the base of the column.

In equation (2) $T.2\pi r$ is the reaction of the tube on the liquid and is equal and opposite to the surface tension pull on the tube.

If $\theta = 0$, $\cos\theta = 1$, and we have

$$h.\rho.g = \frac{2T}{r}.$$

For water, T = 73 dynes per cm. This gives, for a tube 1 mm. bore,

$$h = \frac{146}{980 \times \cdot 05} = 3 \text{ cm. approximately.}$$

Rise of Liquid between Parallel Plates.—Fig. 57 may also be taken to represent two vertical parallel plates dipping into a liquid. The surface of the liquid between them will be cylindrical if they are fairly close together. If d be the distance between the plates, $d = 2R\cos\theta$.

As before, let the pressure at X be P. The pressure at Y will be A, the atmospheric pressure.

We have $$A - P = \frac{T}{R} = \frac{2T.\cos\theta}{d}$$

and $$A = P + h.\rho.g.$$

That is, $$h.\rho.g = \frac{2T.\cos\theta}{d}, \text{ giving } h.$$

Alternatively, we may resolve vertically the forces on the column of liquid contained between the plates and two vertical planes, 1 cm. apart, perpendicular to the plates. We have, then, $2T\cos\theta = h.d.\rho.g$, since the area of the base of the column is d sq. cm., and the forces due to atmospheric pressure are the same at top and bottom.

Example 1.—Two vertical plates dipping in a liquid, whose angle of contact is zero, meet at a small angle θ, show that the line of contact of the liquid on the plates is a rectangular hyperbola.

Let YOX, YOZ be the two planes, Fig. 58. Let P (x, y) be a point on the line of contact of the liquid with YOX.

The horizontal distance between the plates at P is $x.\theta$. But y, the rise of the liquid at this point, is inversely proportional to this distance, and therefore inversely proportional to x. Hence, the curve is a rectangular hyperbola.

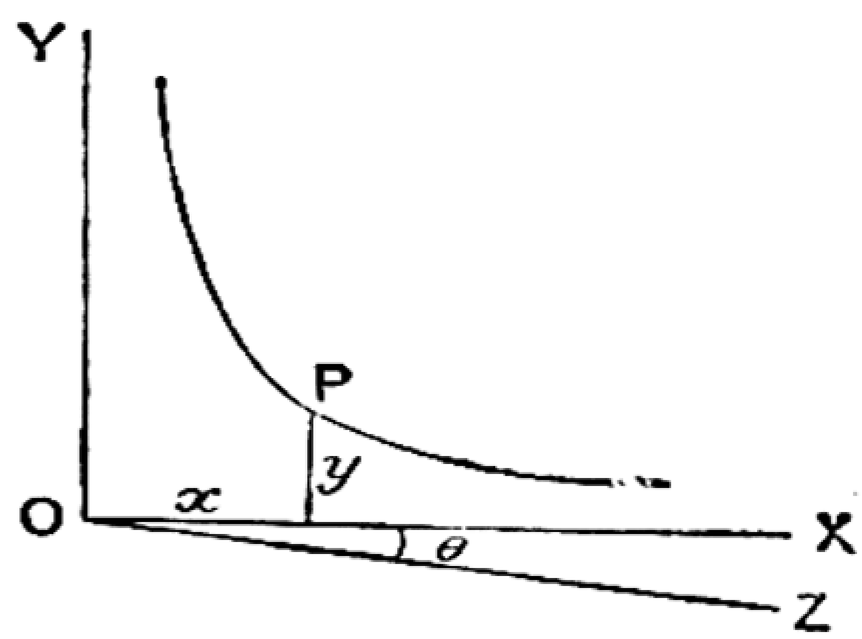

FIG. 58.

Example 2.—What happens if, in Fig. 57, the capillary tube is pushed down into the liquid until the part projecting above the surface is less than h ?

The meniscus will be at the top of the tube, but it remains concave. The angle of contact increases until, when the top of the tube is flush with the liquid outside, it becomes 90°, and the surface of the liquid inside and outside the tube is plane.

9. Methods of Measuring Surface Tension

(1) *Capillary Tube Method.*—For a liquid which wets glass and has zero angle of contact the capillary tube method is convenient. The beaker containing the liquid and the tube should be quite clean and free from any trace of grease. The measurement of the rise in the tube is made easier if a needle is fixed vertically by the side of the tube with its point just on the surface of the liquid in the beaker. The vertical distance from the meniscus to the top of the needle can be measured with a small cathetometer. The length of the needle may be measured afterwards with the same instrument. The radius of the capillary in the neighbourhood of the meniscus has also to be found. This can be done either by breaking the tube at this point and measuring the diameter with the cathetometer, or by filling the tube with mercury for a measured distance near the position of the meniscus. This should be done several times and the mercury collected and weighed.

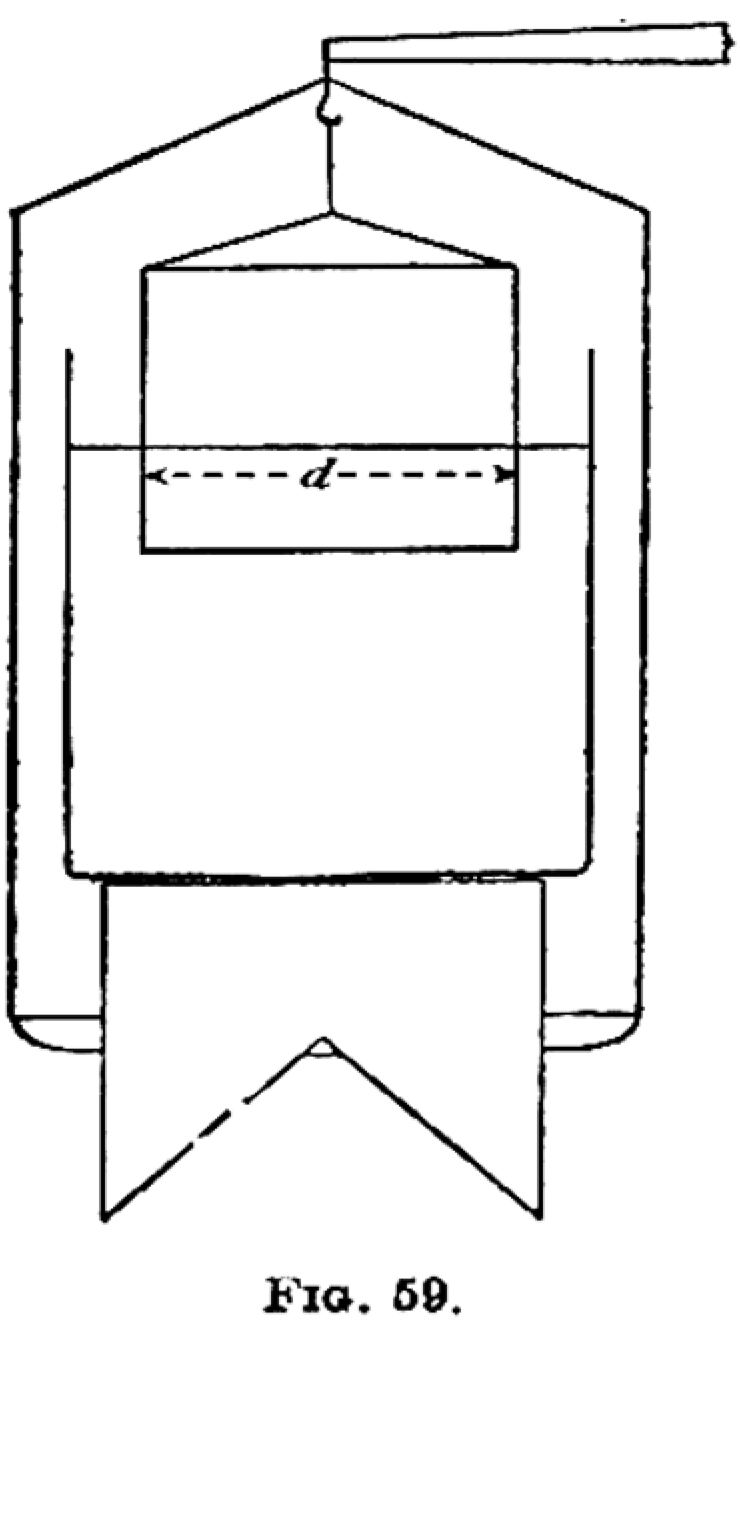

Fig. 59.

The surface tension is given by the formula

$$T = \frac{h.\rho.g.r}{2}.$$

(2) *By Weighing.* — The direct method illustrated in Fig. 59 gives good results with soap solutions, and is possible with tap-water. With tap-water the film lasts long enough to obtain a balance, but some patience is necessary. The rectangular frame of thin wire is dipped into the beaker and a film obtained. Weights to counterpoise the weight of the frame and the pull of the film are placed in the other pan (m_1).

The film breaks, and the counterpoising weight (m_2) is again found. We have

$$2T.d = (m_1 - m_2).g.$$

(3) *Weight of Drop Method.*—This method may be used to compare the surface tensions of liquids which have zero angle of contact with glass.

A short glass tube, with ends squarely cut, is fastened to a wider tube, containing the liquid, by means of a short length of rubber tubing provided with a screw clip. Drops are allowed to form slowly on the end of the glass tube, the outside of which should be wet with the liquid. Just before the drop falls it has the shape shown in Fig. 60. The cylindrical portion narrows into a neck and the drop breaks off. The portion shown in the figure is almost in equilibrium.

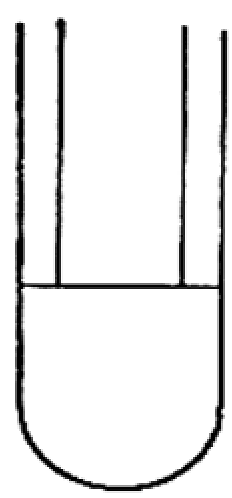

FIG. 60.

The forces on it are $2\pi r.T$ upwards and its weight $m.g$ downwards, r being the external radius of the tube. There are also the forces due to atmospheric pressure, both up and down, and at the end of the tube there is an excess of pressure over atmospheric, due to the cylindrical form, of amount $\frac{T}{r}$.

We have, then,

$$2\pi r.T - \pi r^2.\frac{T}{r} = mg,$$

or

$$\pi.r.T = m.g.$$

This is obviously only an approximate treatment. Results in accordance with experiment are obtained if the numerical factor π is replaced by 3·8. When, however, the method is used for comparing surface tensions, the numerical factor is unimportant. If the same tube is used for different liquids, T is proportional to m, the mass of the drop. A number of drops can be counted as they fall into a beaker and weighed.

If tubes of different diameters are used with a single liquid, the weight of drop will be found to be proportional to the external radius of the tube.

An alternative method is to attach the small tube to a burette (tapless, on account of grease, but provided with a screw-clip) and count the numbers of drops for the same volume of the two liquids. If the surface tensions of the two liquids are T_1 and

T_2, their densities ρ_1 and ρ_2, and the numbers of drops n_1 and n_2, it follows from above that

$$\frac{T_1}{T_2} = \frac{\rho_1}{\rho_2} \cdot \frac{n_2}{n_1}.$$

(4) *Dimensions of a Large Drop.*—For a liquid like mercury the surface tension and angle of contact with a solid can be deduced from measurements of the dimensions of a large drop placed on a flat surface. The shapes of drops of different sizes are shown in Fig. 61. Small drops are nearly spherical. Drops

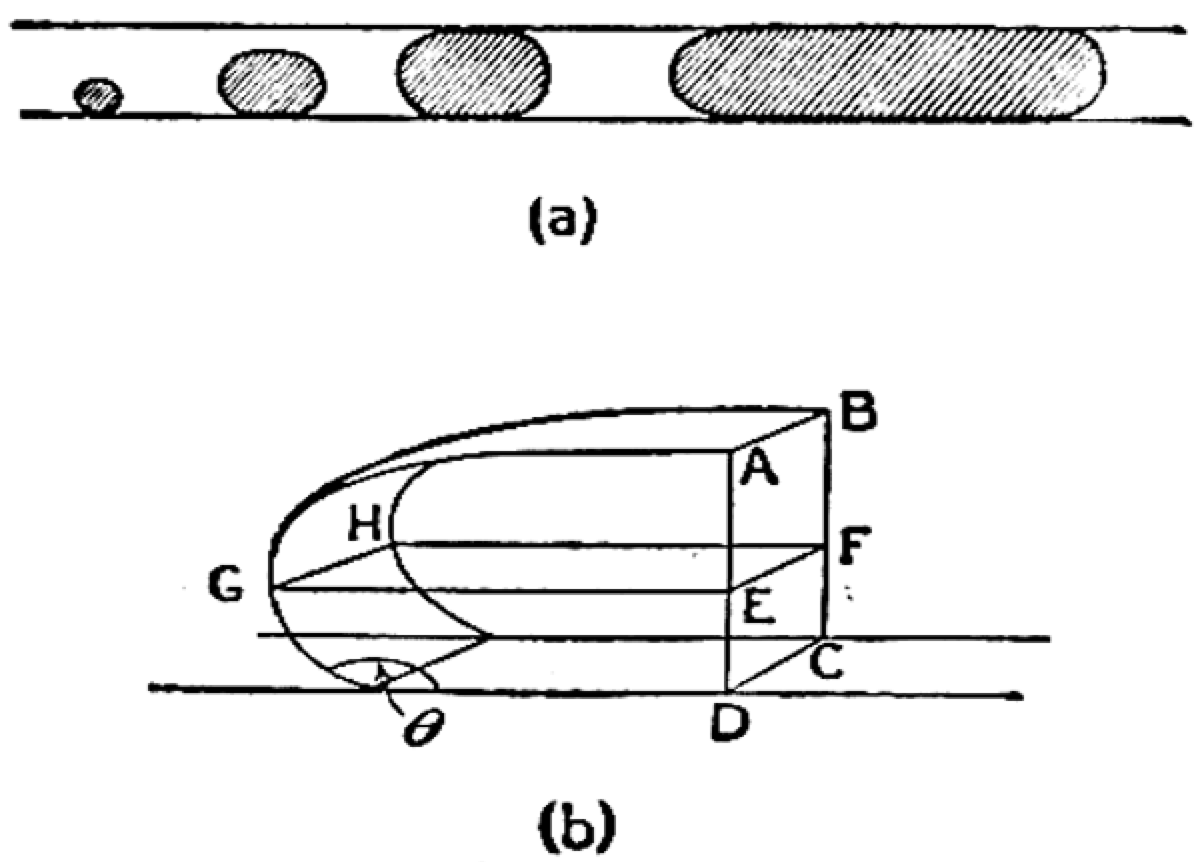

FIG. 61.

which are large enough to be flat at the top all have the same height.

Consider the equilibrium of the portion of the drop contained between vertical planes AGD, BHC, 1 cm. apart, and the perpendicular vertical plane ABCD through the centre of the drop. Let the horizontal plane GEFH cut the drop in the section of greatest area, then the tangent at G to the curve AG is vertical. Denote AE by a, and AD by h. The horizontal forces parallel to the plane of the paper acting on the portion ABFEGH of the drop are T across AB to the right, the force due to atmospheric pressure over the curved surface to the right, the equal force over ABEF due to atmospheric pressure to the left, and the hydrostatic pressure over ABEF to the left.

We have, then, if ρ is the density of the liquid,

$$T = \int_0^a \rho . g . x dx = \rho . g . \frac{a^2}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Similarly, for the horizontal forces on the portion ABCDGH, we have, if θ is the angle of contact,

$$T + T \cos (180 - \theta) = \int_o^h \rho . g . x dx = \rho . g . \frac{h^2}{2} \quad . \quad . \quad (2)$$

Combining equations (1) and (2),

$$1 - \cos \theta = \frac{h^2}{a^2},$$

that is,
$$\cos \theta = \frac{a^2 - h^2}{a^2}.$$

This is negative. For drops of this shape the angle of contact is, of course, greater than 90°.

The drop of mercury is placed on a clean horizontal glass plate, and the measurement of a and h made with a small cathetometer.

The results obtained in surface tension measurements, even by the best workers, differ in some cases by more than 1% from each other. A very slight contamination of the liquid surface may modify the surface tension to this extent.

The surface tensions of a number of liquids and their angles of contact with glass are given in the following table.

Liquid	T dynes per cm.	Angle of Contact
Water	73 at 15° C.	0°
Mercury	547 ,, 17·5° C.	139°
Aniline	43 ,, 15° C.	—
Ethyl Alcohol	22 ,, 20° C.	0°
Methyl Alcohol	23 ,, 20° C.	0°
Ethyl Ether	16·5 ,, 20° C.	16°
Glycerine	65 ,, 18° C.	—
Turpentine	27 ,, 15° C.	17°
Benzene	29 ,, 17·5° C.	0°

10. Measurement of Surface Tension at Various Temperatures

(1) The capillary tube method has been used for determining the surface tension of a liquid at different temperatures.

A convenient form of this method is shown in Fig. 62. The liquid is contained in a U-tube with limbs of different radius.

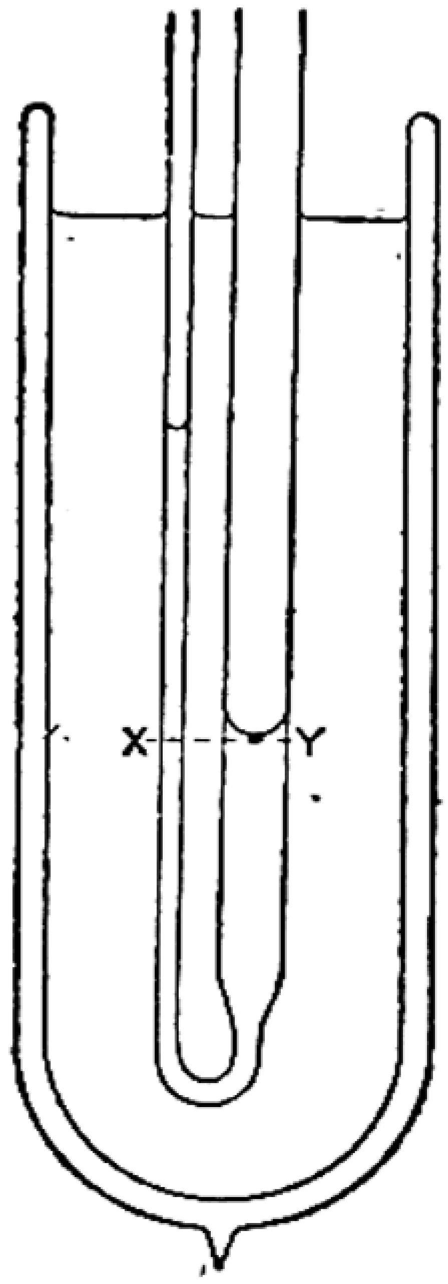

FIG. 62.

This is placed in a non-silvered thermos flask containing water, a heating coil, and a stirrer. For low temperatures alcohol, cooled with carbon dioxide snow, can be used.

Assuming zero angle of contact, we have for the equal pressures at X and Y,

$$A - \frac{2T}{r} + \rho . g . h = A - \frac{2T}{R},$$

where A is the atmospheric pressure, r and R are the radii of the two limbs of the tube, ρ is the density, and h is the difference in level of the liquid in the two sides.

Whence $$2T\left(\frac{1}{r} - \frac{1}{R}\right) = \rho . g . h.$$

To compare the values of T at different temperatures we have to measure only the difference in level, since T is proportional to h.

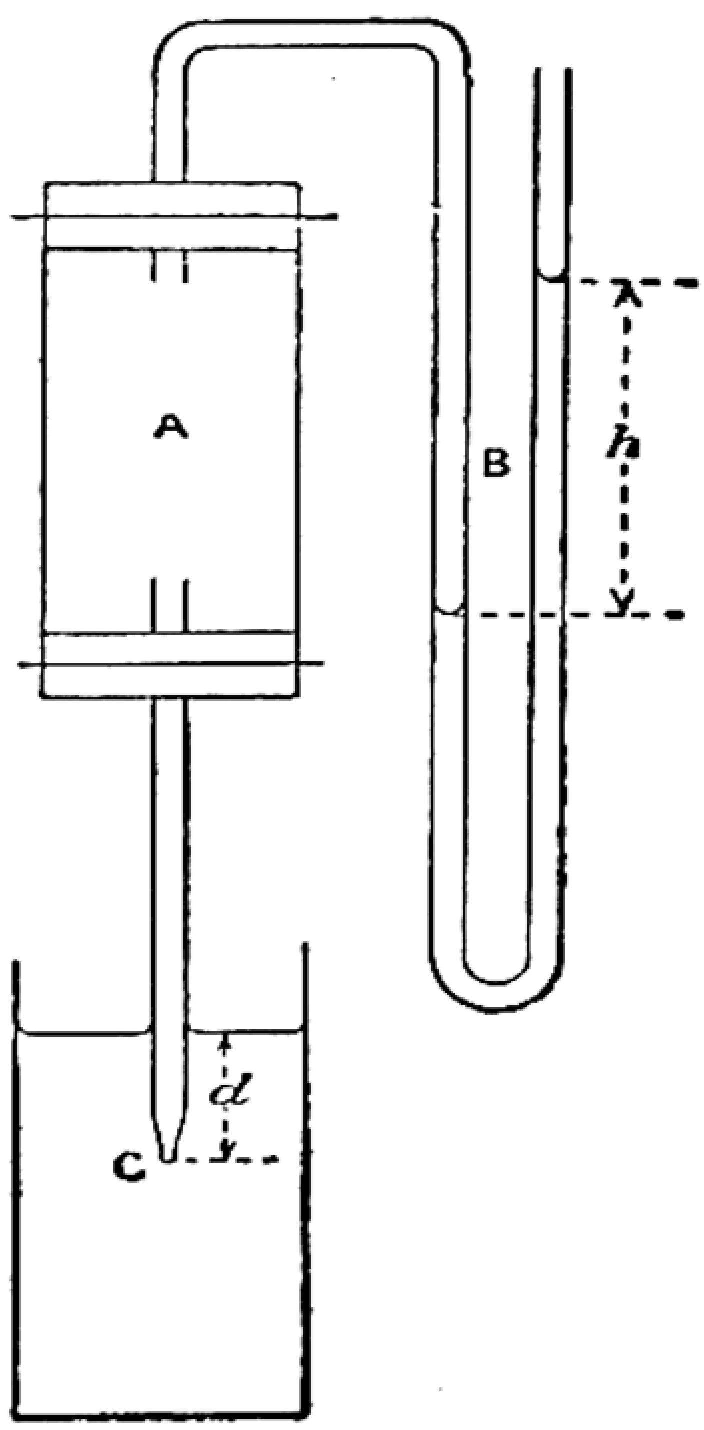

FIG. 63.

(2) Another method is due to Jaeger. A simple form of apparatus is shown in Fig. 63. The idea of the method is to measure the pressure necessary to blow small bubbles in the liquid whose surface tension is being found. A is a glass vessel containing air and a coil of thin wire through which an electric current can be passed. The pressure of the air in A can be raised slightly by warming it in this way, and controlled easily by means of a variable resistance in the circuit. The pressure is measured on the gauge B containing some light, thin oil. C is a capillary jet on the end of which the bubbles are formed. As a bubble is formed the difference in level of the gauge increases to a maximum and then drops slightly as the bubble breaks away. This maximum difference in level occurs when the bubble is hemispherical, its radius at this stage

being a minimum. The corresponding pressure is the sum of that required to form the bubble and that due to the hydrostatic pressure in the liquid at the depth of the jet's orifice. By measuring h and d the pressure required for the bubble can be found. The temperature of the liquid in the beaker is also noted. The surface tension is proportional to the pressure required to form the bubble.

11. Surface Tension of Aqueous Solutions

The surface tension of soap solutions is considerably less than that of water, whilst that of most inorganic salt solutions is slightly greater, increasing with the concentration.

Films from solutions are usually more stable than water films. A water film held in a vertical position breaks very quickly. If the surface tension is the same at the top of the film as at the bottom, there is no force to support the weight of the film. We should, therefore, expect films of pure liquids to be unstable. With films of solutions the concentration of the solute adjusts itself so that the surface tension at the top is slightly greater than at the bottom when the film is held vertically, and the weight of the film can be supported for a considerable time.

12. Surface Tension and Vapour Pressure

When a liquid is confined in a closed vessel containing only the liquid and its vapour, a state of equilibrium is reached in which the same number of molecules enter and leave the surface of the liquid per second. The pressure of the vapour under these conditions is called the saturation vapour pressure of the liquid for the prevailing temperature. This pressure is independent of the size of the vessel and the proportions of liquid and vapour present. It increases with the temperature.

It is clear that this pressure will depend also to a certain extent on the shape of the liquid surface. Over a concave surface we should expect the equilibrium to be reached with fewer vapour molecules per c.c. than over a flat surface, since it is easier for the molecules to return to the liquid. Over a convex surface the number of molecules per c.c., when equilibrium is established, will be greater than over a flat surface. The pressure exerted by the vapour at a particular temperature is proportional to the number of vapour molecules per c.c.

The surface of a small drop is highly convex, and such a drop in an atmosphere of its own vapour at the normal saturation pressure for its temperature, will not be in equilibrium, but will

evaporate, this tendency being greater the smaller the radius of the drop. The equilibrium vapour pressure for a small drop is higher than the normal saturation vapour pressure.

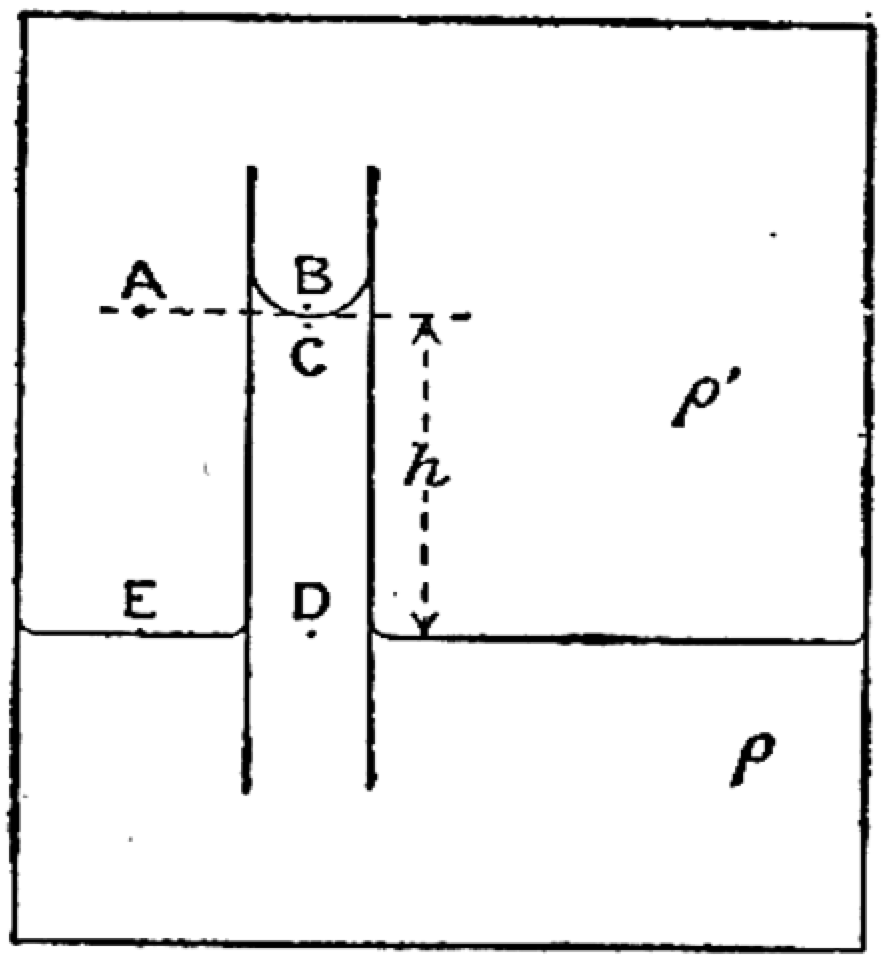

FIG. 64.

To find the relation between the radius of the surface, the change in vapour pressure, and the surface tension of the liquid, suppose we have a closed vessel containing only the liquid, its vapour, and a capillary tube as in Fig. 64. Let ρ be the density of the liquid and ρ' the mean density of the vapour over the height h. Let r be the radius of the tube, supposed vertical. Let the pressure at E and D be p, at A and B, p'.

At C the pressure will be $p' - \dfrac{2T}{r}$, assuming for simplicity a zero angle of contact.

At E, $$p = p' + \rho'.g.h.$$

At D, $$p = p' - \frac{2T}{r} + \rho.g.h.$$

Eliminating $g.h$, we have

$$\frac{p-p'}{\rho'} = \frac{p-p'}{\rho} + \frac{2T}{r.\rho'},$$

that is, $$p - p' = \frac{\rho'}{\rho-\rho'}.\frac{2T}{r} = \frac{\rho'}{\rho}.\frac{2T}{r},$$

if ρ' is small compared with ρ.

p' is the vapour pressure in equilibrium with the concave surface.

If the surface is convex, we have a depression of the liquid in the capillary and p' greater than p; the equation then becomes

$$p' - p = \frac{\rho'}{\rho}.\frac{2T}{r}.$$

If a given volume of air, nearly saturated with water vapour, is subjected to a sudden expansion, it will be cooled and the water vapour present will be more than sufficient, at the lower temperature, to saturate the air. When the expansion is large

the surplus water vapour condenses in the form of a fine mist or cloud. With a smaller expansion it is possible to produce supersaturation of the space occupied, no cloud being formed. The degree of supersaturation that can be obtained is greater if the air is free from dust particles. On the other hand, the cloud is formed readily if the air contains dust particles, or smoke, or ions (charged molecular aggregates). The dust particles or ions serve as comparatively large nuclei on which condensation of the water vapour can begin at a lower vapour pressure than is necessary to form the extremely small initial drops when no dust particles or ions are present.

13. Cohesion Pressure, Latent Heat, and Surface Tension

In section 2 of this chapter the phenomenon of Surface Tension was explained in terms of a force of attraction which the molecules of a liquid are supposed to exert on each other over short distances. In the interior of the liquid the resultant force on an individual molecule, due to the attractions of surrounding molecules, is zero ; but if we consider a plane area in the body of the liquid, the molecules on one side of this plane exert a strong combined attraction on the molecules on the other side. This attraction gives rise to, and is balanced by, an internal pressure called the cohesion pressure.

For a solid the cohesion pressure is equal roughly to the tensile strength of the material and is considerable.

In the case of liquids (and gases) an indication of the magnitude of the cohesion pressure can be obtained from Van der Waals' equation. It is the term $\frac{a}{V^2}$, added to the pressure, in this equation. For water, Van der Waals calculated it to be of the order of 11,000 atmospheres, or, say, $1{\cdot}1 \times 10^{10}$ dynes per sq. cm. This estimate can only be a rough one, as Van der Waals' equation does not represent at all accurately the behaviour of a substance near its critical point, and still less does it do so at temperatures below the critical temperature.

Another estimate of the cohesion pressure can be made from the latent heat of vaporisation of the liquid. The ordinary latent heat of vaporisation consists of two parts : (1) the energy required to separate the molecules of 1 gm. of the liquid against their mutual attractions and disperse them ; (2) the energy or work required to thrust aside the external atmosphere as the vapour is formed. The first part is called the internal latent heat ; the second, the external latent heat. It is the internal

latent heat which is connected with cohesion pressure. According to Laplace's theory the cohesion pressure is equal to half the internal latent heat per unit volume, though it is doubtful if the mathematical method by which this result is obtained is applicable here. It assumes that the range of molecular action is large compared with the size of a molecule, and there are indications that this is not the case.

The latent heat of water at 100° C. is 540 calories. Since 1 gm. of water becomes about 1,700 c.c. of steam at 100° C. and 1 atmosphere pressure, the external latent heat is $1{,}700 \times 10^6$ ergs, or roughly 40 calories. This leaves 500 cals. for the internal latent heat, which gives for the cohesion pressure

$$\frac{1}{2} \times 500 \times 4{\cdot}2 \times 10^7 \text{ dynes per sq. cm., or } 10{,}500 \text{ atmospheres,}$$

taking 1 atmosphere as 10^6 dynes per sq. cm.

The connection between surface tension or surface energy and latent heat may be illustrated by reference to Fig. 48. Let us consider the translation of a molecule from the depth of the liquid to the vapour. No force is required to withdraw the molecule from the liquid until it comes within a distance C of the surface. The force required then increases as the surface is approached and reaches its highest value when the molecule is in the surface. As the molecule is drawn above the surface, the force required decreases and vanishes at a distance C above the surface. At the same distance from the surface, either above or below, the force is the same. It follows, then, that the work done in bringing a molecule into the surface is half that required to evaporate it from the body of the liquid.

It is interesting, in connection with Laplace's theory, to calculate the energy required to form 1 sq. cm. of new liquid surface by bringing molecules to the surface from the interior of the liquid. For mercury the calculation is as follows.

The latent heat of mercury at its B. Pt., 357° C., is 68 calories.

Since 201 gm. of mercury vapour would occupy 22·4 litres at N.T.P. (if it could exist under such conditions), 1 gm. at the B. Pt. and 1 atmosphere pressure will occupy

$$\frac{22{\cdot}4}{201} \times \frac{273 + 357}{273} \text{ litres} = 257 \text{ c.c.}$$

The external latent heat is thus equal to

$$257 \times 10^6 \text{ ergs} = 6{\cdot}1 \text{ cals.}$$

The internal latent heat is, therefore, about 62 cals. Further,

in 201 gm. of mercury there will be $6{\cdot}06 \times 10^{23}$ monatomic molecules (Avogadro's Number).

If w is the work in ergs required to evaporate a single molecule of mercury, we have

$$6{\cdot}06 \times 10^{23} \times w = 201 \times 62 \times 4{\cdot}2 \times 10^7.$$

Whence $$w = 8{\cdot}6 \times 10^{-13} \text{ ergs.}$$

Consider the energy required to form a surface layer 1 molecule thick and 1 sq. cm. in area. To bring each molecule into the surface requires $\frac{w}{2}$ ergs.

At its B. Pt. 1 c.c. of mercury contains 12·74 gms. The number of molecules in one face of a centimetre cube of mercury will therefore be

$$\left(\frac{6{\cdot}06}{201} \times 12{\cdot}74 \times 10^{23}\right)^{\frac{2}{3}} = 11{\cdot}4 \times 10^{14}.$$

The energy required is, therefore,

$$11{\cdot}4 \times 10^{14} \times 4{\cdot}3 \times 10^{-13} = 490 \text{ ergs.}$$

To form a surface layer 2 or 3 molecules thick will require a considerably greater expenditure of energy.

The surface tension of mercury at its B. Pt. is about 410 dynes per cm.

It would appear, then, that the surface layer, of thickness C (the range of molecular action), in which the surface energy resides, is only 1 molecule thick. In Laplace's theory it is assumed that the thickness C contains many layers of molecules.

If this calculation is performed for water or benzine, the energy required to form 1 sq. cm. of surface 1 molecule thick will be found to be about five times as big as the measured surface tension. It should be noticed that we have assumed in the above that the density in the surface layer is the same as in the body of the liquid. This is probably not the case. However, the range of molecular action must be much smaller than is contemplated in Laplace's theory.

More recent experimental work suggests that the peculiarity of the surface layer lies in a definite parallel orientation of the surface molecules, in contrast to the random orientation in the liquid below the surface.

EXAMPLES

1. Define surface tension and deduce an expression for the excess pressure in a soap-bubble of radius r cm. and surface tension T dynes/cm.

Two soap bubbles of unequal diameter are blown at the ends of two tubes which are afterwards put in connection with each other so that there is a free passage for air between the bubbles. Explain what happens. (N.U.)

2. Describe a method of measuring the surface tension of a liquid.

A very small bubble of radius r rises through a liquid of density ρ and surface tension T. If the acceleration of gravity is g and the pressure of the air above the liquid is P, write down an expression for the pressure within the bubble when it is at a depth h below the surface. The temperature remaining constant, what will be the relation between the radius of the bubble and its depth below the surface? (N.U.)

3. What is the ratio of the masses of air inside two soap-bubbles of radii 4 cm. and 6 cm. respectively, if the bubbles are formed in the atmosphere on a day when the mercury barometer reads 75 cm. and are at the same temperature? The surface tension of the soap-film is 30 dynes per cm.

4. Calculate the density of a liquid whose surface tension is 30 dynes per cm., and which rises 3 cm. in a capillary tube whose diameter is 0·6 mm., the contact angle being zero.

5. Using glass capillary tubing, a reading microscope and the ordinary apparatus of a physics laboratory, how would you make an accurate determination of the surface tension between water and glass?

Two plane glass plates, in contact along a vertical line and inclined to each other, are partly immersed in a liquid of density 1·06 gm. per c.c. and surface tension 52 dynes per cm. If the level of the liquid between the plates at 1 cm. from the line of contact is different from that outside by 1 cm., what is the angle between the plates? ($g = 980$, contact angle $= 0°$). (N.U.)

6. Two plane glass plates, fixed at a small uniform distance d apart, are clamped in a vertical position and partly immersed in a liquid of density ρ. Find the height (above the level in the outer vessel) to which the liquid will rise between the plates, if the surface tension is T dynes per cm. and the angle of contact 0°. (N.U.)

7. Give two methods of measuring the surface tension of a liquid. In what way does the surface tension vary with temperature? (C. Schol.)

8. Discuss the rise of a liquid in a capillary tube. A U tube, whose ends are open and whose limbs are vertical, contains mercury. The diameter of one limb is 2 mm., and the diameter of the other is 0·6 mm. Find the difference in level of the mercury surfaces in the two limbs (S.T. = 547, angle of contact = 139°, Sp. gr. = 13·6). (C. Schol.)

9. A disc of thickness 1 mm. and of density 3·5 gm. per c.c. is supported by surface tension at the interface between two liquids of density 1·5 gm. per c.c. Show that the radius of the disc must be less than 1 cm. if the surface tension at the interface is 98 dynes per cm.

10. Give reasons for expecting that the properties of the surface layer of a liquid will be different from the properties of the bulk of the liquid. A piece of cylindrical glass tubing, open at the ends and having a wall 2 mm. thick, is suspended vertically from one arm of a balance. The balance is counterpoised and free to swing. The lower end of the cylinder is then immersed in a vessel containing liquid, and it is seen that when the depth of immersion is 5 mm. the balance remains in equilibrium. Calculate the surface tension of the liquid (contact angle zero; density of liquid 0·8 gm. per c.c.).

11. Explain what is meant by the surface tension of a liquid and describe some well-known phenomena which depend upon this property.

If the surface tension of water is 80 c.g.s. units, find the amount of work required to divide a cubic centimetre of water into a million equal spherical drops.

12. Distinguish between surface tension and surface energy.

A loop of cotton is placed upon a plane soap-film, and the film inside the loop is then broken. Show that the loop will assume a circular form and find an expression for the tension in the cotton in terms of the surface tension of the film and the radius of the loop. (C. Schol.)

[The film will contract to a minimum area, that is, the area inside the loop will be a maximum. For a given periphery, this area is a circle.

Let the radius be r, the surface tension T, and the tension of the cotton F.

Suppose the radius increases to $r + \delta r$.

The total work done is

$$2\mathrm{T}\,.\,\delta(\pi r^2) - \mathrm{F}\,.\,\delta(2\pi r) = 0.$$

That is,
$$2\mathrm{T} \times 2\pi r\,.\,\delta r = \mathrm{F} \times 2\pi\,.\,\delta r,$$

or
$$\mathrm{F} = 2\mathrm{T}\,.\,r.]$$

13. Show that the vapour pressure of a liquid is different for a curved surface and a flat one, and find an expression for the difference. What is the bearing of this result on the phenomenon of supersaturation ?

CHAPTER VII

FRICTION

1. Laws of Friction

A WOODEN block resting on a horizontal table (Fig. 65) is in equilibrium, the weight Mg of the block being balanced by the normal reaction R of the table on the block. If a small horizontal force P be now allowed to act on the block it still remains at rest. We explain this by saying that a friction force F has been called into play equal and opposite to P. If the force P is slowly increased, a point will be reached when the block suddenly begins to move, and a somewhat smaller value of the force will

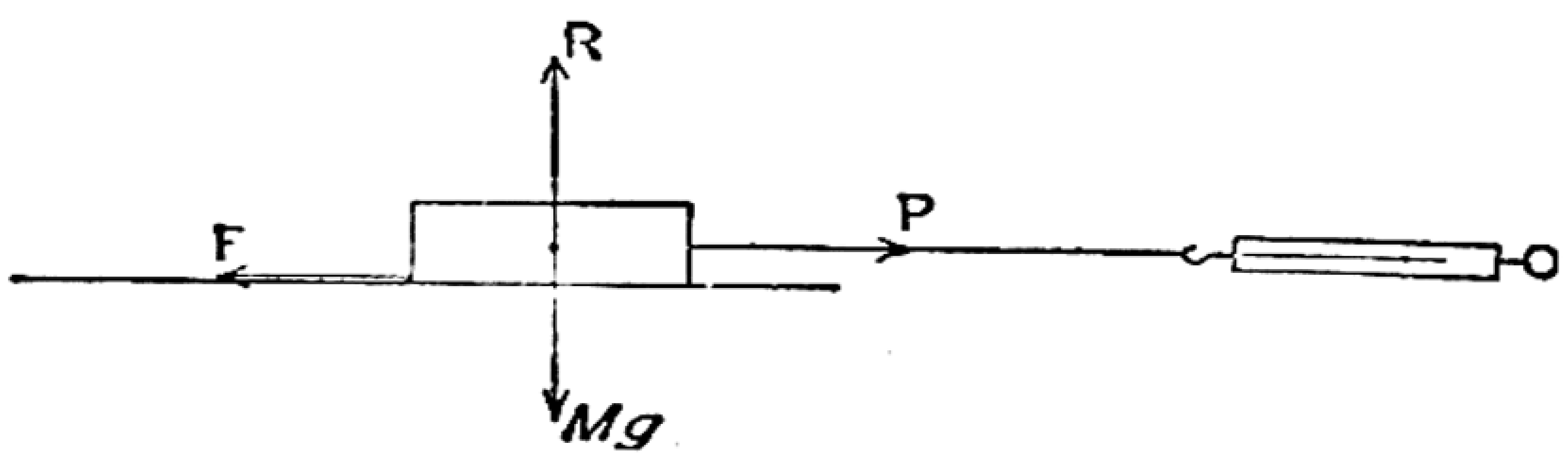

FIG. 65.

then suffice to keep the block in steady motion. The friction force F at all stages in this experiment is equal and opposite to P.

The greatest value of F is called the Limiting Friction. The value of F when the block is in steady motion is the Kinetic Friction.

It is also found by experiment that for two given surfaces in contact the Limiting Friction and the Kinetic Friction are proportional to the normal reaction, and independent of the areas of the surfaces in contact.

Definitions.—The ratio of the limiting friction to the normal reaction for a given pair of surfaces is called their Static Coefficient of Friction.

The ratio of the kinetic friction to the normal reaction is called the Kinetic Coefficient of Friction.

The Kinetic Coefficient is always less than the Static Coefficient.

The above experimental facts are summed up in the Laws of Friction. These laws have a purely experimental basis, and, for dry surfaces, are only roughly true.

(1) When two surfaces in contact are at rest relatively to one another the friction force between them is just sufficient to prevent motion.

(2) There is a limit to the amount of the friction force that can be called into play.

(3) This limiting friction is proportional to the normal reaction between the two surfaces, but is independent of the area of contact. The coefficient of friction depends only on the nature of the surfaces.

(4) When the surfaces are in relative motion, the friction force between them (the kinetic friction) is proportional to the normal reaction, but independent of the area of contact and the speed of relative motion. It acts on either surface in such a direction as to oppose their relative motion.

2. Experimental Verification

These laws may be verified, and the coefficients of friction for two surfaces roughly measured, by means of the apparatus suggested in Fig. 65. The force P can be applied and measured by the spring-balance. The force P is equal to the friction force F. Different weights can be placed on the block in order to vary the normal reaction R. It is convenient to use for this experiment a block with top and bottom faces of different area. By repeating the measurements of P with the block turned over, it may be verified that the friction is independent of the area of contact.

It is worth while to do this experiment carefully in order to obtain an idea of the degree of accuracy of the Laws of Friction.

3. Angle of Friction

In Fig. 66, which represents a heavy particle at rest on a rough horizontal surface, P is an applied force and F is the friction called into play and preserving equilibrium. F and R are really the components of a single force S, which is the resultant reaction of the surface on the particle. If θ is the angle between S and the normal to the surface,

$$\tan \theta = \frac{F}{R}.$$

Now the limiting friction (which is the greatest value F can have) equals $\mu.R$, where μ is the static coefficient of friction.

So $$\tan\theta = \frac{F}{R} \leqq \mu.$$

The angle λ, such that $\tan\lambda = \mu$, is the greatest possible value of θ, and is called the angle of friction. It is the angle between the normal and the reaction of the surface when the friction is limiting.

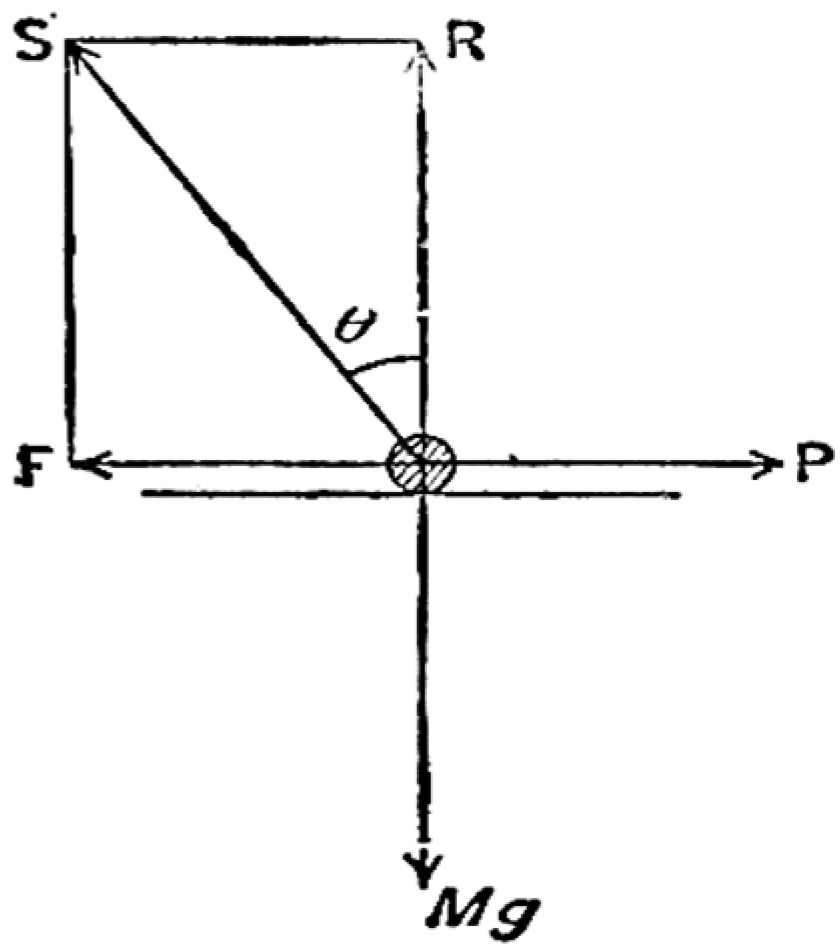

FIG. 66.

4. Particle Resting on a Rough Inclined Plane

Let the coefficient of friction be μ.

The friction force F will act up the plane and will be just sufficient to preserve equilibrium. Resolving parallel and

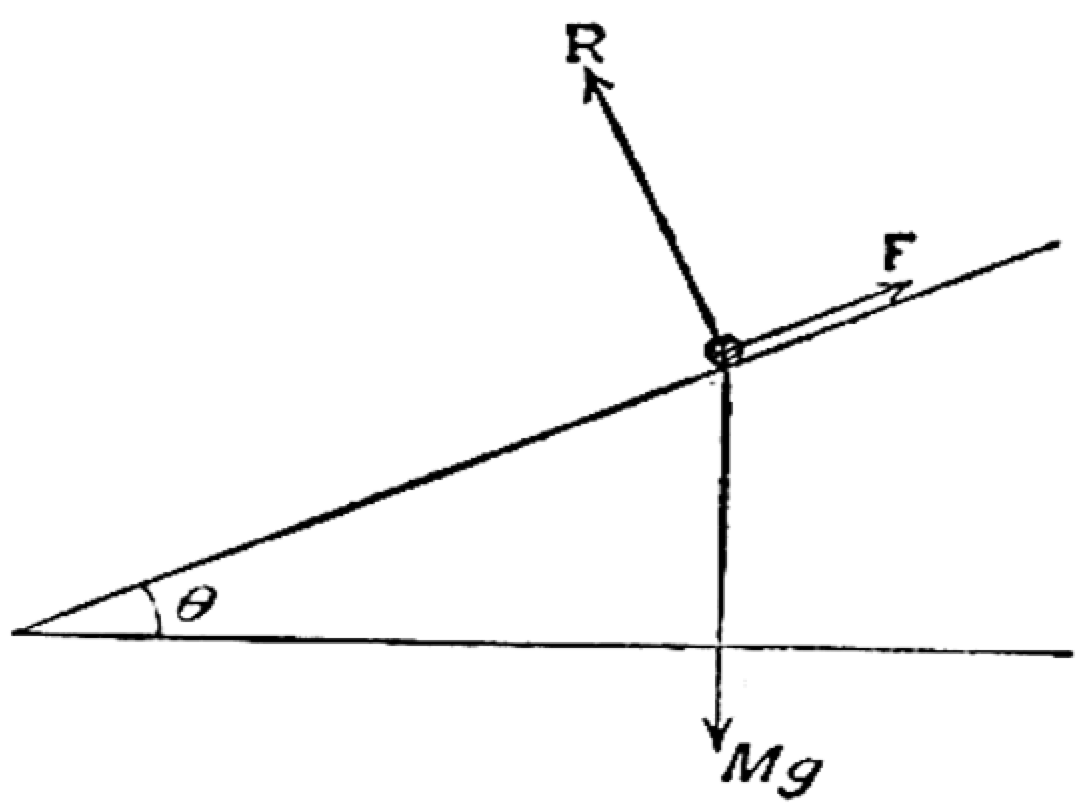

FIG. 67.

perpendicular to the plane, we have

$$F = Mg\sin\theta$$

and $$R = Mg\cos\theta,$$

that is, $$\frac{F}{R} = \tan\theta.$$

But $F \leqq \mu R$, that is, $R\tan\theta \leqq \mu R$, or $\tan\theta \leqq \mu$.

Thus for equilibrium to be possible

$$\tan\theta \leqq \tan\lambda,$$

where λ is the angle of friction.

This suggests another method of finding the static coefficient of friction. The block of Fig. 65 may be placed on the plane which is tilted slowly until sliding begins. If the inclination of the plane to the horizontal is then θ, $\mu = \tan\theta$.

Note.—In statical problems involving friction the student should guard against making the assumption that the friction force is equal to $\mu . \mathrm{R}$, unless it is quite clear from the question that the friction is limiting.

Example 1.—Suppose, in the above, $\tan\theta > \mu$, and we have to find the least force P, parallel to the plane, to prevent the particle sliding down.

The particle is on the point of sliding down, the friction is therefore limiting and equal to $\mu . \mathrm{R}$, and acts up the plane. By resolving parallel and perpendicular to the plane, P may be found in terms of $\mathrm{M}g$, μ, and θ.

If we require the least force P to move the particle up the plane, we note again that the friction is limiting, but now acts down the plane.

Example 2.—Find the least force, P, required to drag a particle of mass M along a rough horizontal plane.

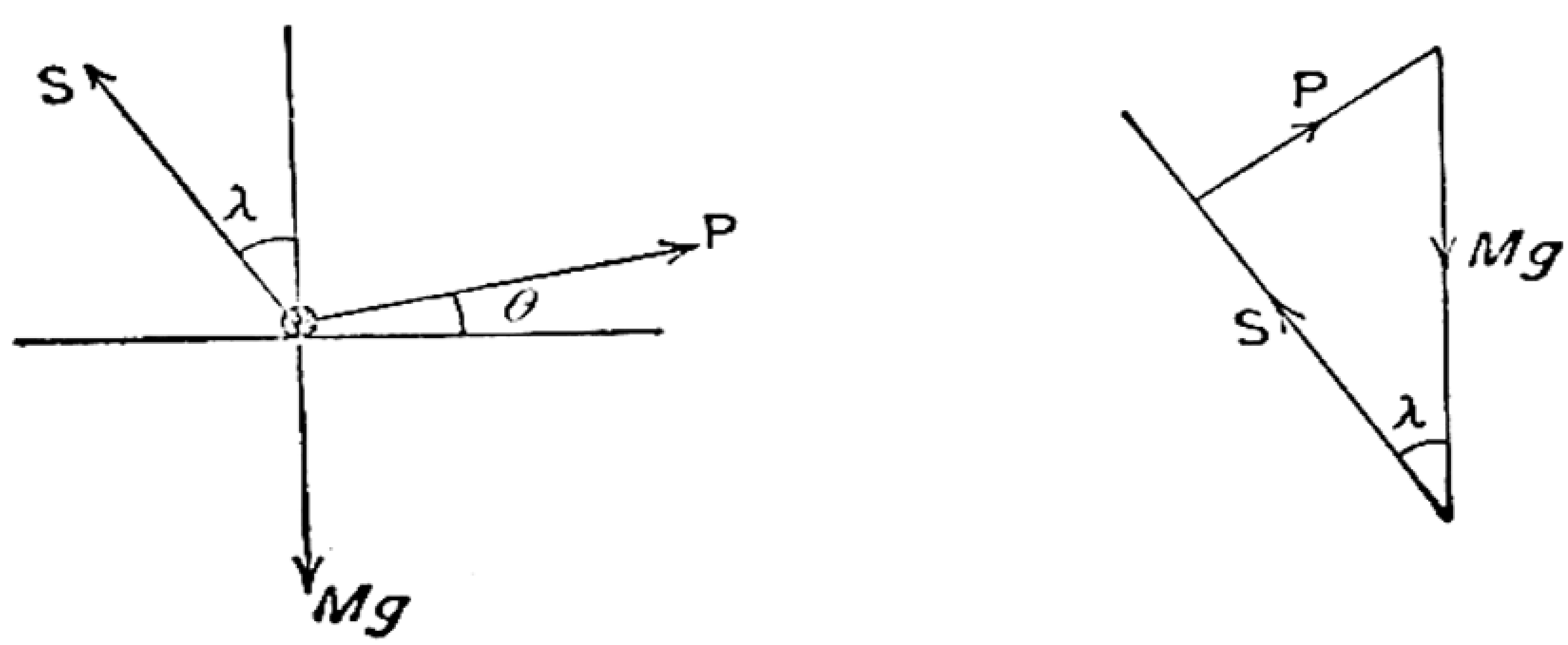

Fig. 68.

Let the coefficient of friction be μ, and let P make an angle θ with the horizontal, Fig. 68.

The friction is limiting, and the reaction S of the plane on the particle makes an angle λ with the vertical, where λ is the angle of friction and $\tan\lambda = \mu$. Three forces only act on the particle. If we draw the triangle of forces, it is clear that P is least when it is perpendicular to S. That is, when $\theta = \lambda$ and $\mathrm{P} = \mathrm{M}g \sin\lambda$.

Note that the horizontal force required to move the block is $\mu . \mathrm{M}g = \mathrm{M}g . \tan\lambda$; $\tan\lambda$ is, of course, greater than $\sin\lambda$.

The analytical solution of this problem is as follows.

Resolving horizontally and vertically, we have

$$P\cos\theta = S\sin\lambda$$

and $$P\sin\theta + S\cos\lambda = Mg.$$

That is, $$P\sin\theta + P\cos\theta.\cot\lambda = Mg \quad . \ . \ . \ . \ (1)$$

To find the minimum value of P we differentiate and put $\frac{dP}{d\theta} = 0.$

$$\frac{dP}{d\theta}.\sin\theta + P.\cos\theta + \frac{dP}{d\theta}.\cos\theta\cot\lambda - P\sin\theta\cot\lambda = 0.$$

That is, $$P.\cos\theta = P.\sin\theta.\cot\lambda,$$

or $$\theta = \lambda.$$

Multiplying (1) by $\sin\lambda$, $(= \sin\theta)$,

we have $$P\sin^2\lambda + P\cos^2\lambda = Mg\sin\lambda,$$

that is, $$P = Mg.\sin\lambda.$$

5. Acceleration of a Particle down a Rough Inclined Plane (see Fig. 67)

Unless $\tan\theta > \mu$, the particle will remain at rest (μ, here, being the static coefficient of friction). Suppose $\tan\theta > \mu$. Let the acceleration of the particle down the plane be a. We have, then,

$$M.a = Mg\sin\theta - F$$
$$= Mg\sin\theta - \mu R,$$

where μ, in this equation, is the kinetic coefficient of friction.

Also, since the acceleration of the particle perpendicular to the plane is zero,

$$0 = R - Mg\cos\theta.$$

Therefore, $$Ma = Mg\sin\theta - \mu Mg\cos\theta,$$

and $$a = g(\sin\theta - \mu\cos\theta).$$

If the length of the plane is l and the particle starts from rest at the top, the velocity at the bottom is given by $V^2 = 2al$.

The kinetic energy at the bottom

$$= \frac{1}{2}M.V^2 = M.a.l$$

$$= Mgl\sin\theta - \mu Mgl\cos\theta.$$

$Mgl\sin\theta$ is the kinetic energy the particle would have at the bottom of a smooth plane of the same inclination and length. The energy converted into heat, etc., through friction is

$\mu Mgl \cos \theta$. This is equal to the work required to drag the particle horizontally through a distance equal to the projection of the incline on the horizontal.

6. The Band-brake. Measurement of Brake H.P.

In the band-brake a flexible band is pulled tightly round a metal drum fixed to the wheel. The band presses normally on the drum, and if the wheel is in motion there is a tangential force due to friction resisting the rotation. The retarding effect of this force is proportional to its moment about the axle. It is larger in proportion to the radius of the drum.

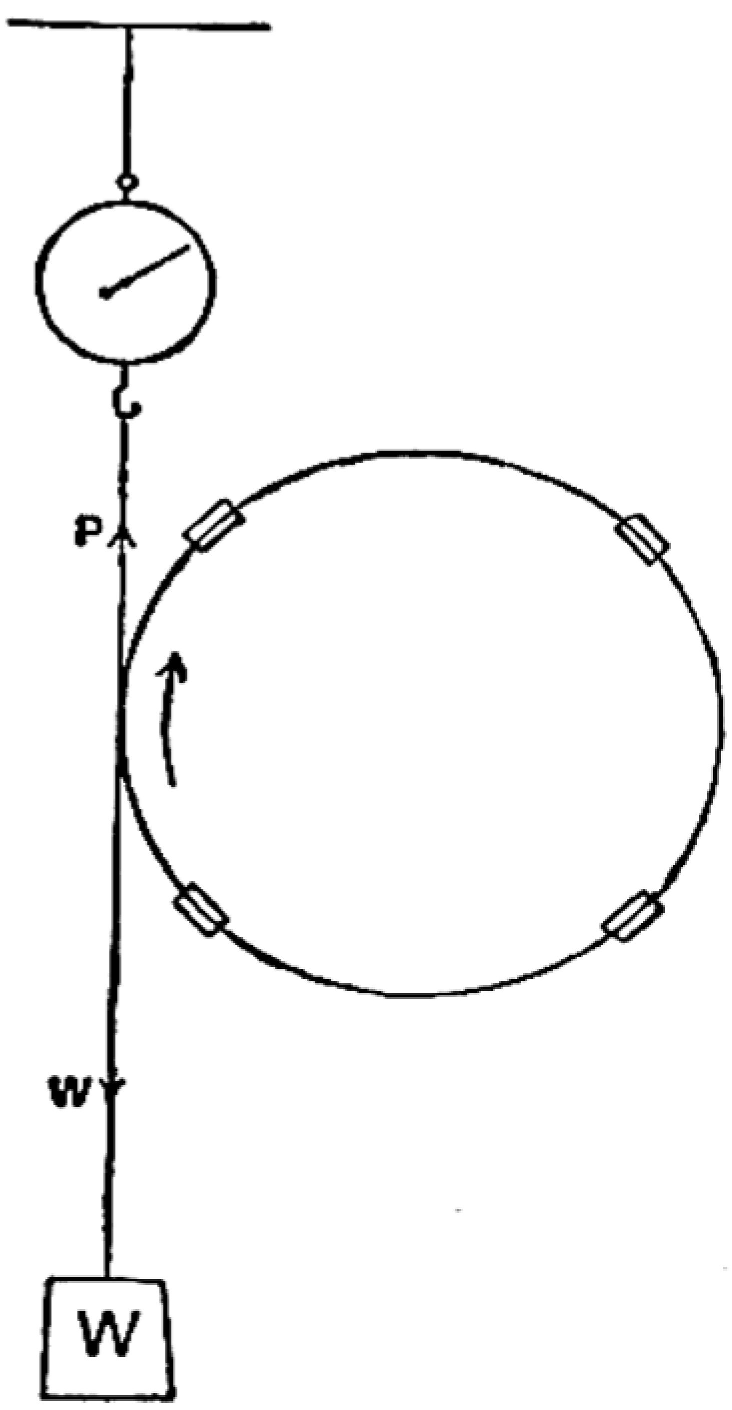

FIG. 69.

A similar device is used by engineers for measuring the brake horse-power of engines, that is, the power transmitted to the axle. A rope is passed round the fly-wheel, one end of the rope being attached to a weight W and the other end to a fixed spring balance. Four wood blocks, attached to the rope, fit loosely round the rim of the fly-wheel and keep the rope in position. The direction of rotation is as shown in Fig. 69. If P is the reading of the spring-balance when the fly-wheel is turning steadily through n revs. per min., the tangential friction force resisting motion is W — P. The moment of this about the centre of rotation is $(W - P).r$, where r is the radius of the wheel. (This is measured to the centre of the rope.) The H.P.

$$= \frac{(W - P).r.2\pi n}{33{,}000},$$

if W and P are in lb. wt. and r in ft.

7. Calculation of the Coefficient of Friction in the Band-brake

Let the band be in contact with a stationary cylinder of radius a, over an arc AB subtending an angle α at the centre O, as in the section shown in Fig. 70.

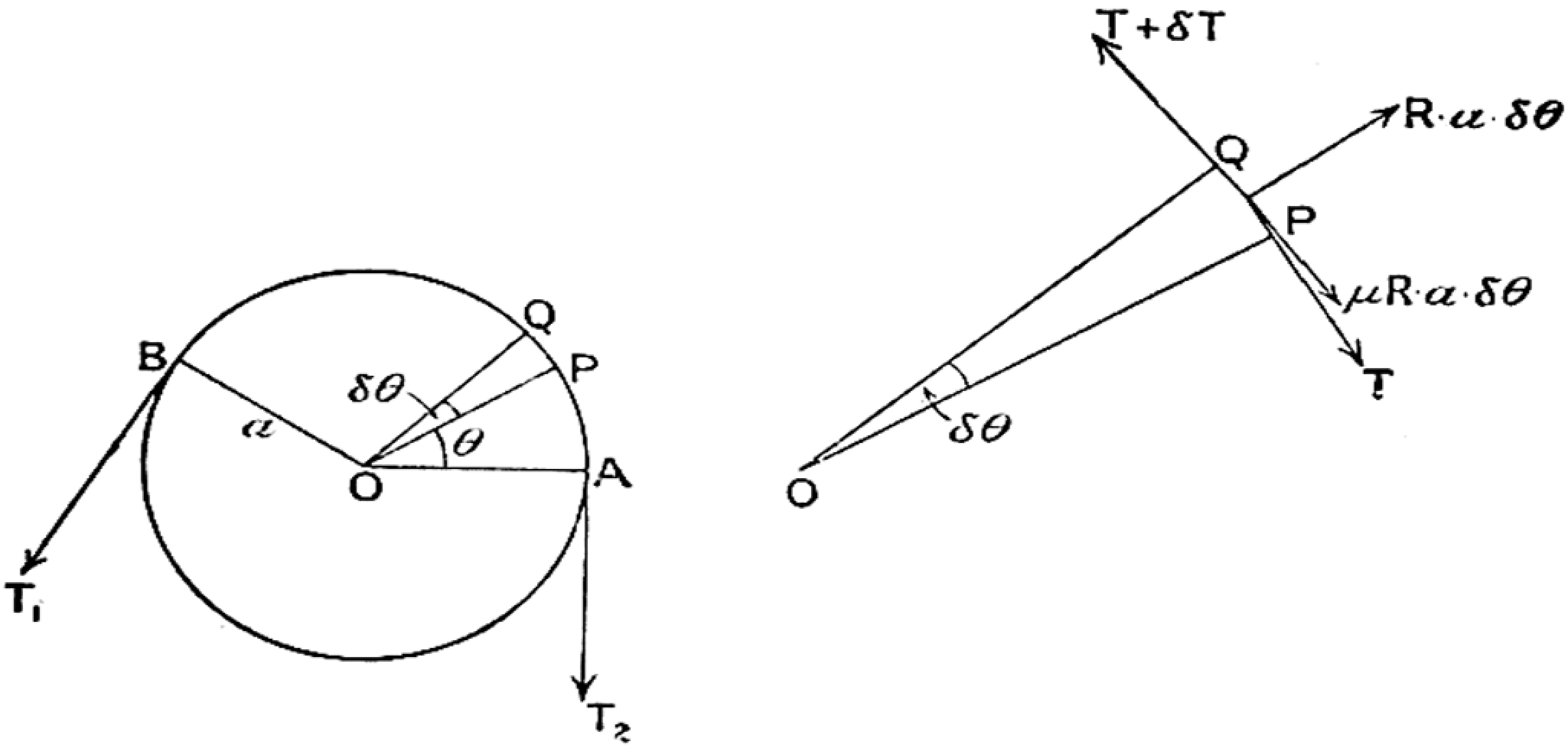

FIG. 70.

Let the tensions of the band at A and B be T_2 and T_1 respectively, T_1 being greater than T_2.

Suppose the band to be on the point of slipping. Consider the forces on the element PQ, of the band, subtending an angle $\delta\theta$ at O.

There is a normal thrust from the cylinder $R.a.\delta\theta$, the thrust being R per unit length. There are tensions T at P and $T + \delta T$ at Q; there is also a tangential friction force $\mu R.a.\delta\theta$.

Resolving tangentially we have

$$\delta T = \mu R.a.\delta\theta, \quad \ldots\ldots \quad (1)$$

and, radially, $$2T \sin\frac{\delta\theta}{2} = R.a.\delta\theta,$$

or $$T = R.a \quad \ldots\ldots \quad (2)$$

From (1) and (2) $$\delta T = \mu.T.\delta\theta.$$

Whence $$\int_{T_2}^{T_1}\frac{dT}{T} = \mu\int_0^\alpha d\theta,$$

that is, $$\log\frac{T_1}{T_2} = \mu\alpha \text{ or } T_1 = T_2.e^{\mu\alpha}.$$

T_1 and T_2 can be measured and μ found from this relation.

For a rope wrapped twice round a cylinder $\alpha = 4\pi$; if $\mu = \cdot 5$ and $T_2 = 10$ lb. wt., then $T_1 = 10.e^{2\pi} = 5{,}320$ lb. wt. The other end of the rope can sustain a pull of more than 5,000 lb. wt. before slipping occurs.

8. Rolling Friction

Consider a sphere, weight W and radius a, resting on a horizontal plane, and acted on by a small force P, as in Fig. 71. A friction force F, equal to P, will be called into play, and the sphere should roll under the action of the couple (P, F), of moment $P.a$, even when P is very small. As a matter of experience, a certain minimum force P is required to cause the sphere to roll. Both the sphere and plane are, in the case of real objects, slightly deformed, and in contact over a small area.

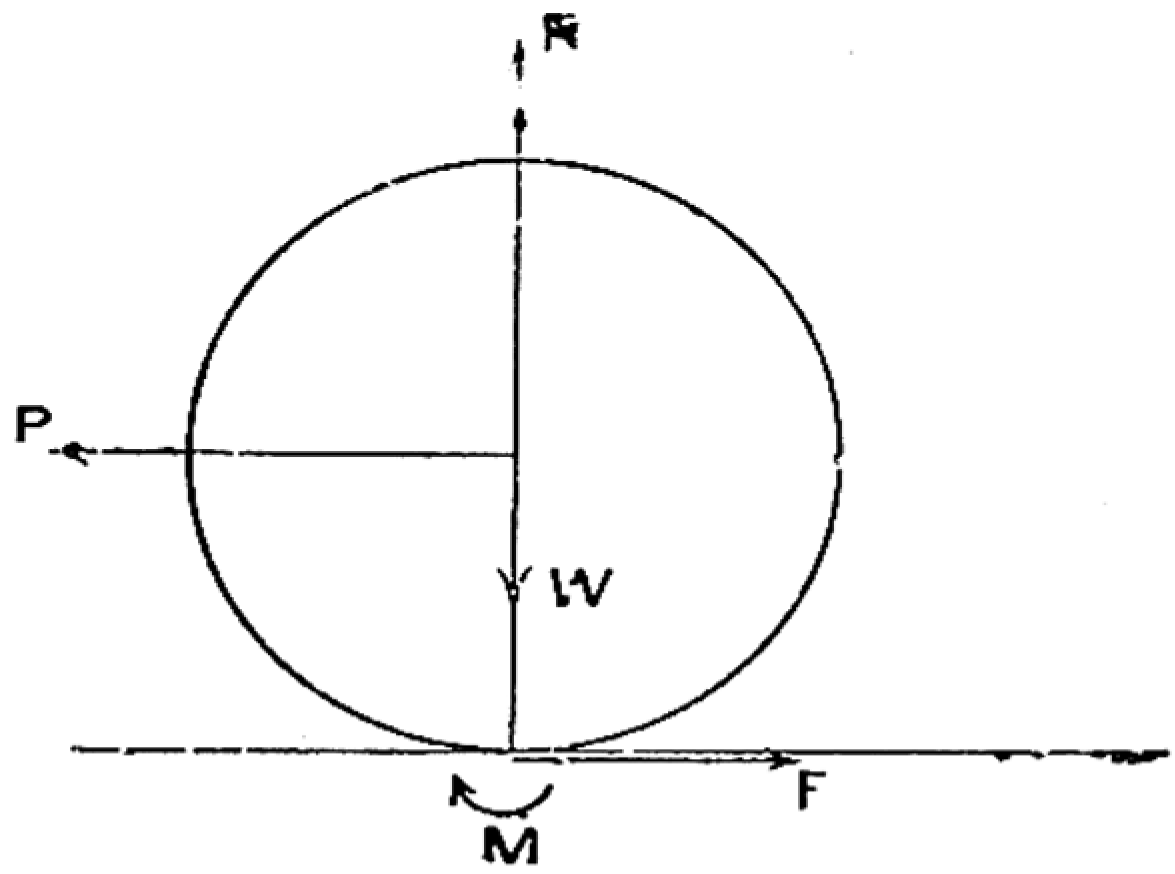

FIG. 71.

(They depart, to a greater or less extent, from our definition of rigid bodies.) It is usually assumed that the forces across this area are equivalent to a couple whose moment M is proportional to the normal reaction R. If we put $M = R.k$, k is a constant depending on the hardness of the surfaces and the radius of the sphere.

In Fig. 71, if the sphere is on the point of rolling,

$$P.a = M = W.k,$$

that is

$$P = \frac{W}{a}.k,$$

and since k is small in the case of hard surfaces, P will also be small.

In questions in which the deformation of the surfaces is neglected, *i.e.*, in which the bodies are rigid, the couple M is not introduced, and the smallest force is sufficient to cause rolling.

9. Lubrication

It is important in engineering practice to reduce the friction between surfaces which have to slide over each other to the lowest possible amount. Power is wasted in overcoming friction, heat is steadily generated, and if the surfaces are dry they will reach a high temperature, suffer abrasion, and may even cohere or 'seize.'

By interposing between the surfaces a suitable lubricant, the coefficient of friction can be reduced from about ·3 for comparatively smooth metal surfaces, to as low a value as ·0005. Since the energy wasted is proportional to the coefficient of friction, lubrication is clearly important.

Most lubricants are liquid, though graphite in the solid form, and graphite greases are used for some purposes. In the case of solid lubricants, the particles of the lubricant either act as rollers between the metal surfaces, or form surfaces which are abraded instead of the metals. The properties of a fluid lubricant which are of importance are its viscosity and its power of maintaining a film over the metal surface. Obviously, also, it should not undergo decomposition at the temperatures to which it is exposed, and it must have no corrosive action on the metal surfaces. These properties are possessed in a high degree by mixtures of mineral and vegetable oils.

For the best results the sliding surfaces must be completely separated by the lubricant. This is secured by providing the bearing with a copious supply of oil, and, where the load on the bearing is heavy, the oil is forced in under pressure. The ordinary cylindrical bearing is easier to lubricate successfully than a piston in its cylinder or flat surfaces sliding over each other. In the cylindrical bearing, as the axle rotates, it is continually dragging oil into the bottom of the bearing, where the metal surfaces are closest together, and so helps to maintain an unbroken layer of lubricant.

When the sliding surfaces are completely separated by a fluid lubricant the frictional resistance to their relative motion depends on the viscosity, or internal friction, of the lubricant, and the laws of friction given at the beginning of this chapter do not apply. The frictional resistance is now proportional to the speed of the relative motion.

If the quantity of lubricant between the surfaces is insufficient to provide a complete film, a condition which may be called partial lubrication, the friction is, of course, greater than with flooded surfaces, but still much less than with dry surfaces. In the case of partial lubrication the laws of friction apply as for dry surfaces. Experimental results with partially lubricated surfaces are more consistent than with dry surfaces, and the laws are more accurately obeyed.

Examples

1. A block of wood is sent sliding along the horizontal surface of a long table. It comes to rest in 2 secs. after sliding a distance of 4 metres. Find the coefficient of friction between the surfaces of the block and the table. (N.U.)

2. Describe experiments to illustrate the laws of friction.

A body resting on a horizontal plane is pulled by a string inclined at 30° to the horizontal. If the coefficient of friction is $\frac{1}{4}$, find the ratio of the weight of the body to the tension when the body is just slipping. (O. and C.)

3. Investigate the relation between the 'coefficient of friction' and the 'angle of friction.'

A uniform cylinder of radius r and height h is placed with its plane base on a rough inclined plane and the inclination of the plane to the horizontal is gradually increased. Show that the cylinder will topple over before it slides if $2r/h$ is less than the tangent of the angle of friction. (N.U.)

4. A man, weighing 140 lb., climbs up a uniform ladder, 20 ft. long and 70 lb. in weight, which rests against a rough vertical wall at an angle of 45°. If the coefficient of friction at each end of the ladder is 0·5, how far will the man be able to climb up the ladder before it begins to slip? (N.U.)

5. Explain what is meant by 'limiting friction.' A uniform rod of weight W and length l rests on a rough horizontal table, the coefficient of friction being μ. A gradually increasing horizontal force is applied perpendicularly at one end of the rod. Assuming that the vertical reaction is distributed uniformly along the rod, show that the rod begins to turn about a point distant $l/\sqrt{2}$ from the end at which the force is applied, and find the magnitude of the applied force. (N.U.)

6. Distinguish between static and sliding (kinetic) friction and define the coefficient of sliding friction.

How would you investigate the laws of sliding friction between wood and iron?

An iron block, mass 10 lb., rests on a wooden plane inclined at 30° to the horizontal. It is found that the least force parallel to the plane which causes the block to slide up the plane is 10 lb. wt. Calculate the coefficient of sliding friction between wood and iron. (N.U.)

7. One end of a rod rests against a rough wall (coefficient of friction μ) at an angle ϕ with the vertical, while the other end is supported by a string fastened to the wall vertically above the

point at which the rod touches it, making an angle θ with the vertical. Find a relation between ϕ and θ when the rod is about to slip.

8. State the laws of sliding friction.

An irregular body is drilled so as to move without play up and down a vertical rod. If the length of the hole is b, and the coefficient of friction μ, find the least distance of the centre of gravity of the body from the centre of the rod so that the body will remain stationary in any position on the rod. (O. Schol.)

9. If μ is the coefficient of friction, show graphically or otherwise when jamming will begin when a drawer with two handles is pulled out by a pull on one handle only. (O. Schol.)

10. State the laws of solid friction.

Find the horizontal force required to push a body of mass M lb. up an inclined plane whose angle of inclination with the horizontal is α, the coefficient of friction between body and plane being μ.

11. Define the 'angle of friction,' and prove that the least force required to drag a body up a rough inclined plane is inclined to the plane at the angle of friction. (O. Schol.)

12. A motor-car has a wheel-base of 10 ft., and when it rests on level ground the centre of gravity is 5 ft. behind the front wheel centres and 3 ft. above the ground. If the coefficient of friction between wheels and road is 0·4, what is the greatest incline on which it can stand without slipping,

(1) when the rear wheels are braked,

(2) when the front wheels are braked? (C. Schol.)

13. State and illustrate the laws of friction.

Find the minimum speed at which a motor cyclist could maintain himself on the vertical wall bounding the outside of a circular track if the radius of curvature of the wall were 160 ft. and the coefficient of friction between the tyres and the wall were 0 8. What would be the slope of his cycle to the vertical? (C. Schol.)

CHAPTER VIII

ELASTICITY

1. Behaviour of a Wire under Various Loads

For this investigation the method suggested in Fig. 72 is in some ways more convenient than the usual pair of vertical wires carrying respectively a scale and a vernier. The wire under test, about 3 metres long, is fixed at one end to a firm support and attached at the other end to the hook of a spring-balance. The ring of the spring-balance is fastened to the end of a threaded bolt which can be moved parallel to its length by means of a wing-nut. The support and the carrier for the bolt are clamped to the table. The extension of the portion of the wire between

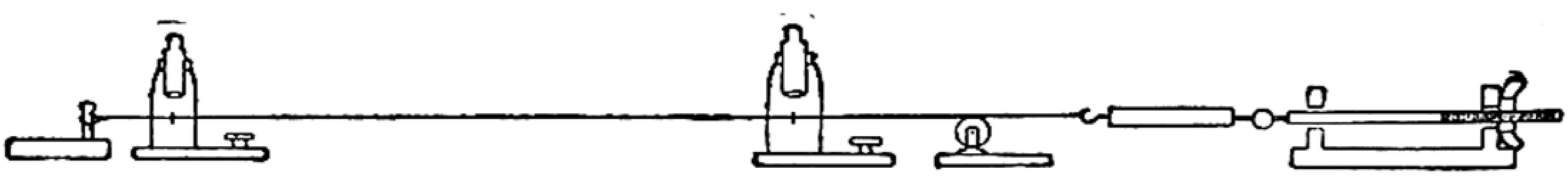

Fig. 72.

two light scratches is measured by two travelling microscopes. The load is applied by means of the screw and nut and read on the spring-balance. A small grooved pulley near the spring-balance keeps the wire steady and makes the observations easy. A definite tension, sufficient to keep the wire taut, is applied and the positions of the scratches noted. The tension is then increased by regular steps. In the early stages extension is proportional to load so the extension due to the initial load can easily be calculated.

2. Elastic Limit. Yield Point. Hooke's Law

A typical load-extension graph for a stretched wire is shown in Fig. 73. Up to the point A the graph is a straight line passing, of course, through the origin. For greater loads the graph bends over, the extensions produced by succeeding equal increments of the load becoming larger. As the load is gradually increased, a point B is reached eventually at which it is impossible to maintain the load steady without continually turning the wing-nut. In

the case of a vertical wire loaded with weights, the extension at this point ceases to be independent of time and increases slowly under constant load. This point is called the Yield Point. At this stage the wire is being drawn out into one or more narrow necks and further addition to the load causes breakage. The point A on the curve is called the Elastic Limit. Up to this point the behaviour of the wire is an example of Hooke's Law :—

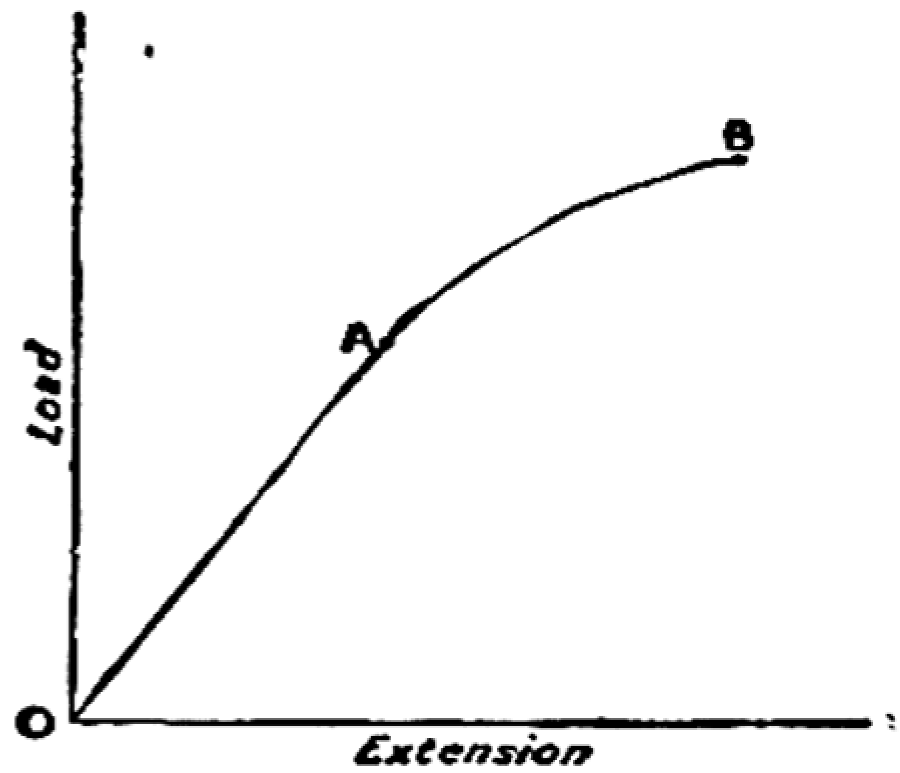

FIG. 73.

The extension or displacement is proportional to the force.

A body or system is said to be Elastic when it obeys Hooke's Law.

Effect of Unloading and Reloading

Fig. 74 shows the effect of unloading the wire gradually. If the unloading is started before the Elastic Limit is reached the straight part of the curve is retraced and the wire returns to its original length. If the wire is loaded up to the point C, between the Elastic Limit and the Yield Point, on unloading the extensions follow the straight line CDE parallel to AO, and the wire has received a permanent extension OE. If it is now loaded up again the graph follows the course EDF, the straight portion being longer than on the original loading. The Elastic Limit has been raised by over-straining the wire.

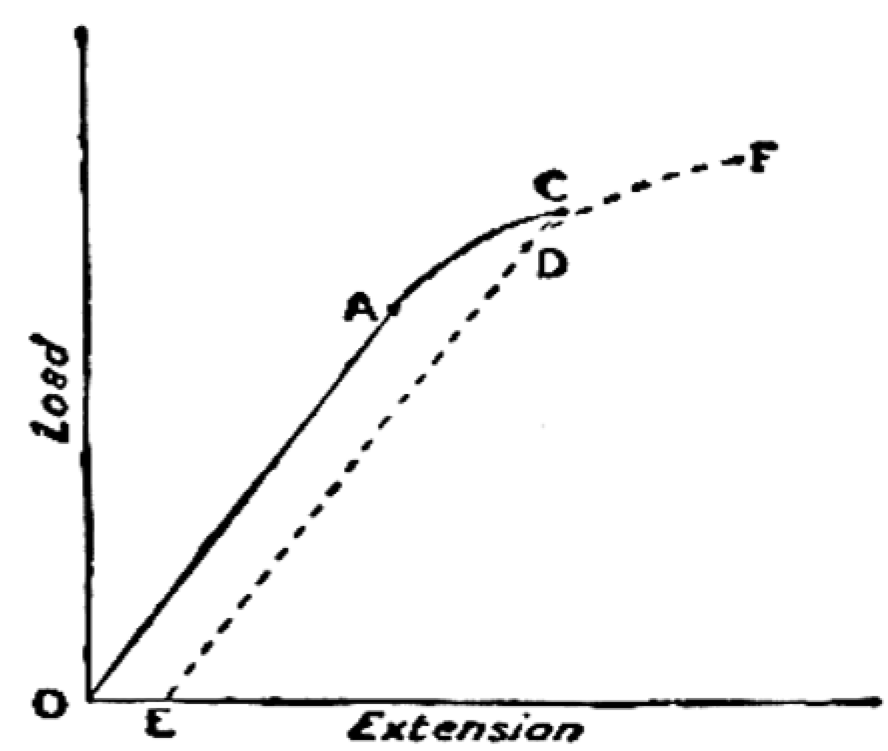

FIG. 74.

3. Stress and Strain. Young's Modulus. Poisson's Ratio

In experiments like the above the longitudinal force applied to the wire divided by the area of cross-section (the force per unit area) is called the Stress.

The extension per unit length of the wire is called the Strain.

The ratio $\frac{\text{Stress}}{\text{Strain}}$, with the above meanings, is called Young's Modulus of Elasticity for the material. In this definition it is understood that the material is not stressed beyond its elastic limit.

When a longitudinal force is applied to a wire or rod a lateral contraction is produced in addition to the extension. The ratio of the lateral contraction per unit of lateral dimension to the strain is called Poisson's Ratio, commonly denoted by σ.

For a cylinder of length L and radius R, which under longitudinal stress undergoes an extension l and a lateral contraction in the radius of r, Poisson's ratio would be $\frac{r/R}{l/L}$.

If the volume of the cylinder remained unchanged we should have,

$$\begin{aligned}\pi R^2 L &= \pi(R - r)^2(L + l)\\ &= \pi R^2 L\left(1 - \frac{r}{R}\right)^2\left(1 + \frac{l}{L}\right)\\ &= \pi R^2 L\left(1 - \frac{2r}{R} + \frac{l}{L}\right),\end{aligned}$$

since $\frac{r}{R}$ and $\frac{l}{L}$ are both small.

Thus $\frac{2r}{R} = \frac{l}{L}$ or $\frac{r/R}{l/L} = \frac{1}{2}$.

For most isotropic materials Poisson's ratio is less than $\frac{1}{2}$ and there is an increase of volume under this kind of stress.

4. Young's Modulus by Experiment on a Wire

The load-extension graph obtained in the experiment described in section 1 can be used to calculate Young's Modulus for the wire. The additional measurements required are the length of the wire between the scratches and the mean diameter of the wire. The load and extension corresponding to any point on the straight portion OA, Fig. 73, may be read from the graph. If E is Young's modulus, P the load, a the area of cross-section of the wire, L the length of the wire and l the extension, $E = \frac{P/a}{l/L}$. Young's modulus is expressed in a variety of units, dynes per sq. cm., gm. wt. per sq. cm., lb. wt. per sq. in., and others. Its dimensions are those of stress, *i.e.*, $\frac{\text{Force}}{\text{Area}}$; the strain has no dimensions, being in this case the ratio of two lengths.

If a rod is subject to compression along its length, its sides being free from constraint, the contraction l produced is also given by the formula $E = \frac{P/a}{l/L}$, provided that the rod is not strained beyond its elastic limit.

The type of stress considered in defining Young's modulus is not the simplest from a theoretical point of view, but it is of great importance in engineering. Engineers measure Young's modulus and the Tensile Strength (Breaking Stress) of materials by experiments on test-pieces in the form of cylindrical rods. Heavy loads are applied by means of a system of massive levers, and the small extensions are measured on the microscope scale of a Ewing extensometer. Experiments on wires are not a reliable guide to the behaviour of a material in bulk. The elastic properties are generally altered by the drastic process of drawing the material into a wire.

Tensile Strength or Breaking Stress or Maximum Stress or Ultimate Stress as it is variously called is defined as the load required to break the specimen divided by the original area of cross-section. For a material which 'necks' before breaking this is less than the actual maximum stress. The Ultimate Stress as defined above, however, is the stress of interest to engineers.

The Ductility of a material also enters into engineering specifications. It is measured by the ultimate percentage elongation of the specimen. For different steels it varies between 10% and 40%. Ductility is often tested roughly by a bending test. A rectangular strip of the material is bent in the middle, and the angle through which the halves are bent when fracture first occurs on the outer surface of the bend is a measure of the specimen's ductility.

Two specimens of steel may have practically the same elastic limit and ultimate stress and yet differ greatly in an important quality designated Toughness. This quality is tested by some form of Impact Test. In the Izod Impact Testing Machine a heavy pendulum is released from a fixed height and strikes the specimen to be broken at the bottom of the swing. The specimen is firmly clamped in a vertical position on the base of the machine. The energy absorbed by the impact is deduced from the angle of the half-swing immediately following the blow, and is taken as a measure of the Toughness of the material. The specimens for this test are made to a standard shape and size in the form of a rectangular bar with a V-notch of specified dimensions across the middle of one face. Fracture occurs at the notch.

The above definition of Ultimate Stress assumes a steadily increasing load. When a material is subjected to Alternating Stresses fracture may occur for stresses considerably less than the Ultimate Stress.

5. The Bulk Modulus of Elasticity

Consider a rectangular block of solid isotropic material exposed to uniform pressures of increasing magnitude. Each dimension of the block is reduced slightly in proportion to the pressure, so long as the elastic limit is not passed, and the volume is decreased. The fractional contraction is the same for each dimension and the block remains the same shape. The system of forces over the faces of the block connoted by a uniform pressure is easy to produce experimentally. In defining the Bulk Modulus, however, we have to suppose these forces reversed and the faces of the block to be acted on by a uniform tension P per unit area outwards at right-angles to the faces. If v is the small increase in volume produced, and V, the original volume, the strain is defined in connection with this modulus as $\frac{v}{V}$, and the stress is P, the force per unit area. The ratio $\frac{P}{v/V}$ is called the bulk modulus of elasticity for the material. It is usually denoted by the symbol k, and sometimes called the Volume Elasticity.

The Compressibility of a substance is the fractional decrease in volume divided by the increment of pressure. It is the reciprocal of the bulk modulus.

The bulk modulus is difficult to measure, but it can be calculated from the values of Young's modulus and the modulus of rigidity. Its value is given by $k = \frac{nE}{9n - 3E}$, where E is Young's modulus, and n the modulus of rigidity.

6. Stress and Strain in a Shear. Modulus of Rigidity

In a shear the stress may be considered to act tangentially. In Fig. 75 ABCD represents a rectangular block. Suppose tangential forces T per unit area are applied at the top and bottom faces as shown. These alone, constituting a couple, would merely produce rotation. They can be balanced by forces T per unit area on the faces AD and BC, acting as in the diagram. The moments of the two couples are clearly equal. This system of forces constitutes a shearing stress. It produces

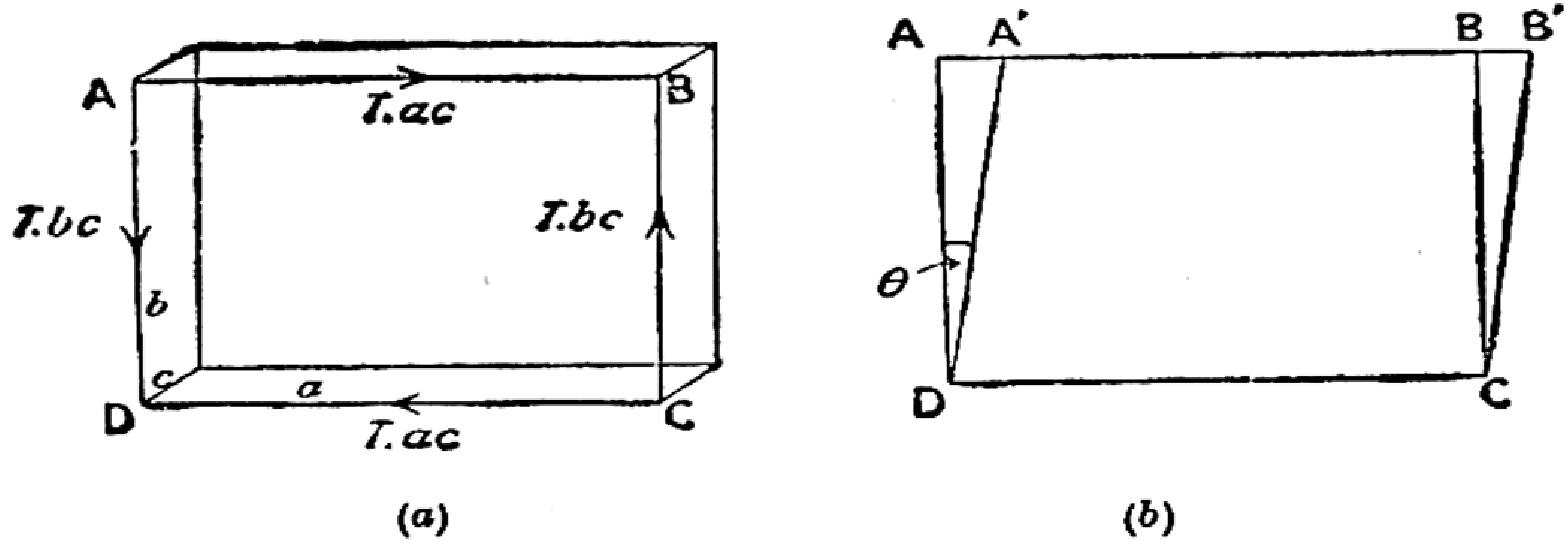

FIG. 75.

a slight deformation of the block, the rectangle ABCD becoming a parallelogram, the volume of the block remaining constant.

The stress in this case is the tangential force per unit area, T. The strain is the angle of shear, θ in Fig. 75, measured in radians. The Rigidity Modulus is defined as $\frac{T}{\theta}$, and is commonly denoted by the symbol n.

7. Torsion of a Cylinder

If equal and opposite couples are applied at the ends of a cylinder in planes perpendicular to the axis, the cylinder is twisted, the particles of each layer being displaced slightly relative to their neighbours in contiguous layers. The displacement is of the nature of a shear, and it is possible to calculate the angle of twist produced by a given couple in terms of the dimensions of the cylinder and its modulus of rigidity.

Consider first a thin cylindrical tube, length l and radius r, coaxial with and forming part of the cylinder. This is represented in Fig. 76. Let it be fixed at its upper end and a couple be applied at the lower end. Let the generating line AB be twisted into the position AC. The angle of twist is θ and the angle of shear is ϕ. Each bit of the thin tube has been sheared through the angle ϕ. If l is large compared with r, ϕ is small and is given by $BC = l\phi = r\theta$.

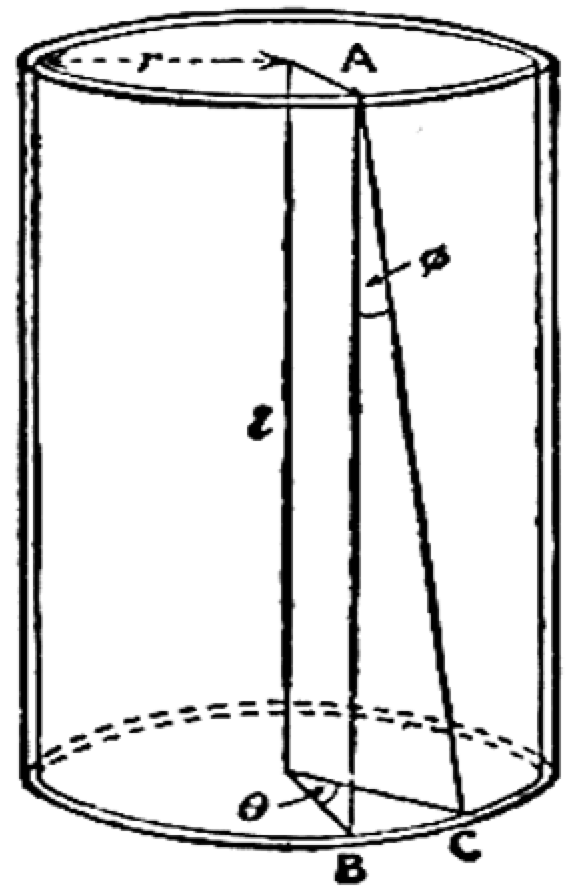

FIG. 76.

Let T be the tangential force per unit area

acting on the base of the tube, then $T = n\phi$, where n is the modulus of rigidity. The total force acting on the base

$$= T . 2\pi r . \delta r,$$

where δr is the thickness of the tube.

The moment about the axis

$$= T . 2\pi r^2 . \delta r$$
$$= n\phi . 2\pi r^2 . \delta r$$
$$= \frac{n\theta . 2\pi r^3 . \delta r}{l}.$$

This gives the couple required to twist the tube through an angle θ radians.

The couple C required to twist a cylinder of length l and radius R is the sum of the couples required for the cylindrical tubes of which it is composed.

Thus $$C = \int_0^R \frac{n\theta . 2\pi r^3 . dr}{l} = \frac{\pi n R^4}{2l} . \theta.$$

For a given cylinder or wire θ is proportional to C. The couple required to produce a twist of 1 radian is known as the torsion constant of the wire. Its value is $\frac{\pi . n . R^4}{2l}$.

It may be mentioned, too, at this point, that it is the modulus of rigidity which is involved in the extension of a helical spring. The wire of which the spring is composed does not increase in length, but is twisted as the spring is extended.

8. Measurement of Modulus of Rigidity

(*a*) *Statical Method.*—For a wire the modulus of rigidity may be measured, and Hooke's Law verified for this type of strain, with the apparatus depicted in Fig. 77. The wire AB is soldered in a metal block at A and carries a pulley at its lower end by means of which twisting couples of various strengths may be applied. The angle of twist for the length AC of the wire is measured by a pointer clamped to the wire at C and a circular degree scale carried by the supporting stand. To verify Hooke's Law a series of different equal weights are placed on the scale pan attached to the pulley and the corresponding angles of twist observed. If w represents the weight and θ the angle of twist, $\frac{w}{\theta}$ will be found constant.

To find the rigidity, the radius of the pulley, the length AC of the wire, and the diameter of the wire must also be measured. The last measurement must be made with the greatest care. From the previous section the rigidity $n = \frac{2lC}{\pi R^4 \theta}$. Since the expression for n involves R^4 an error of 1% in R will produce an error of 4% in n. The diameter of the wire should be measured at several points of its length in two perpendicular directions with a micrometer screw-gauge. To give n in absolute units, l and R must be in centimetres, θ in radians, and C in dyne-centimetres.

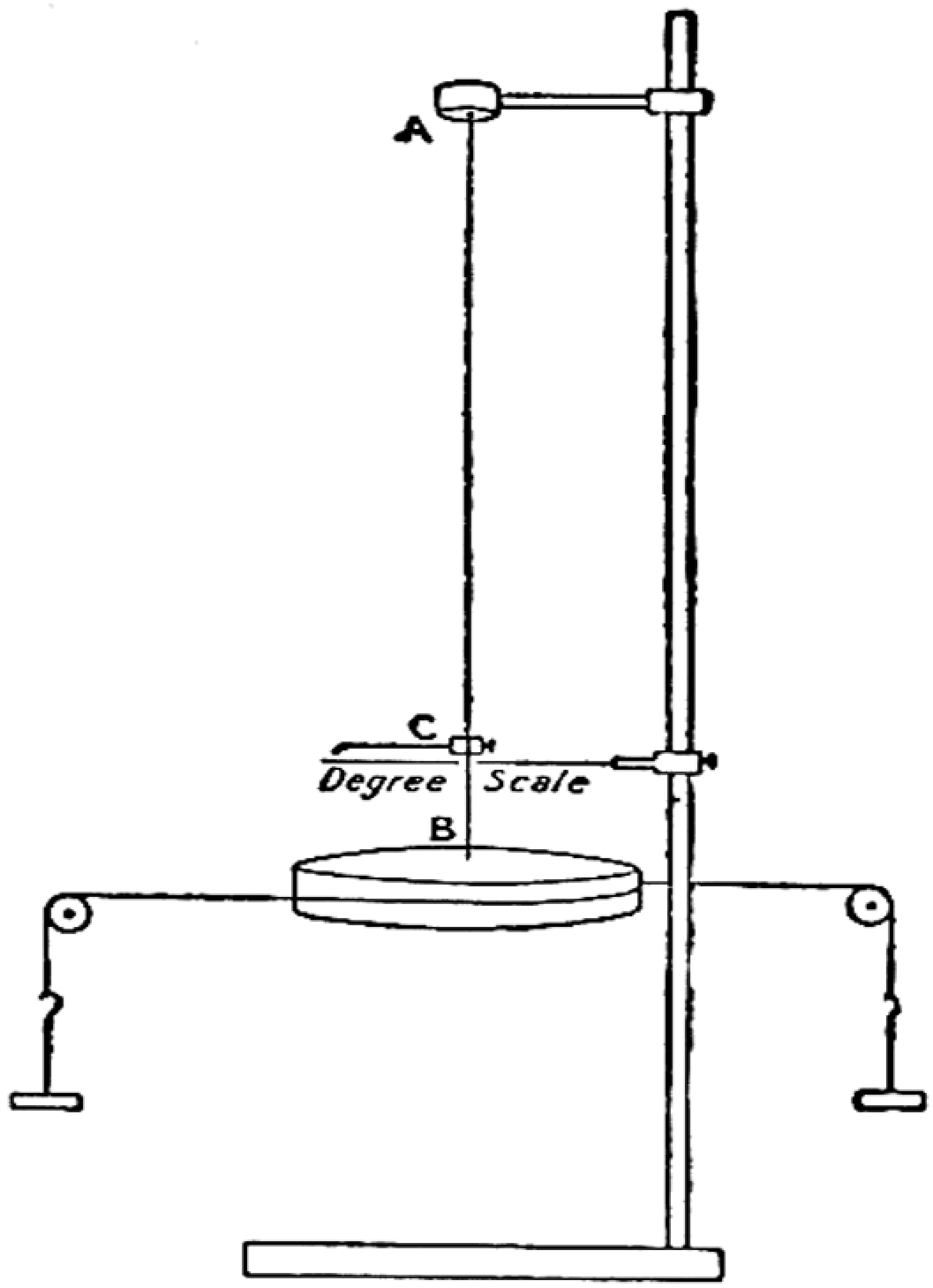

FIG. 77.

Fig. 78 indicates how the measurement of rigidity for a cylindrical rod may be made. The rod is clamped at one end and supported in ball-bearings near the other. The twisting moment is applied by a cord (carrying a heavy weight) fixed to a pulley wheel on the end of the rod. The angle of twist is

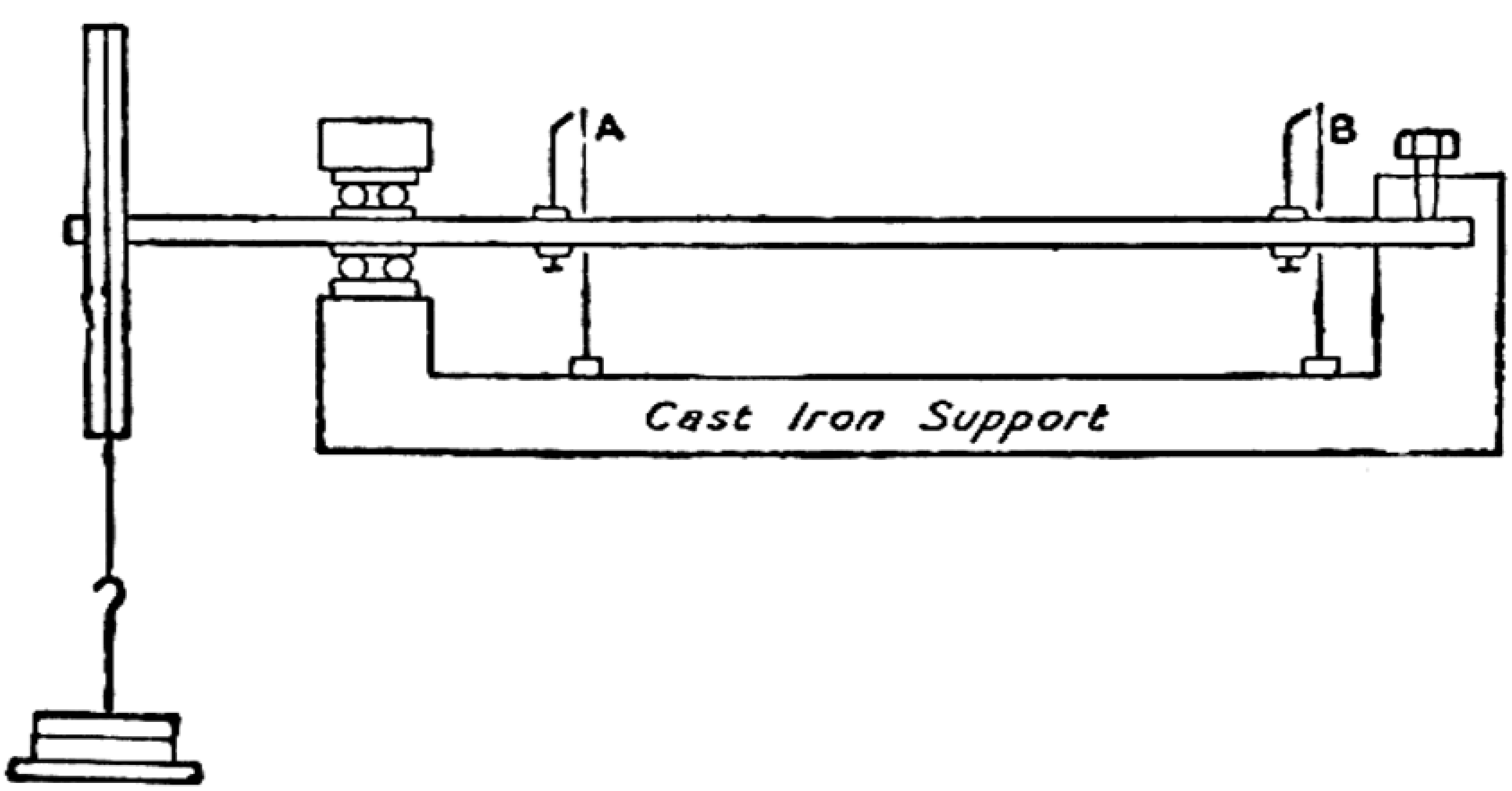

FIG. 78.

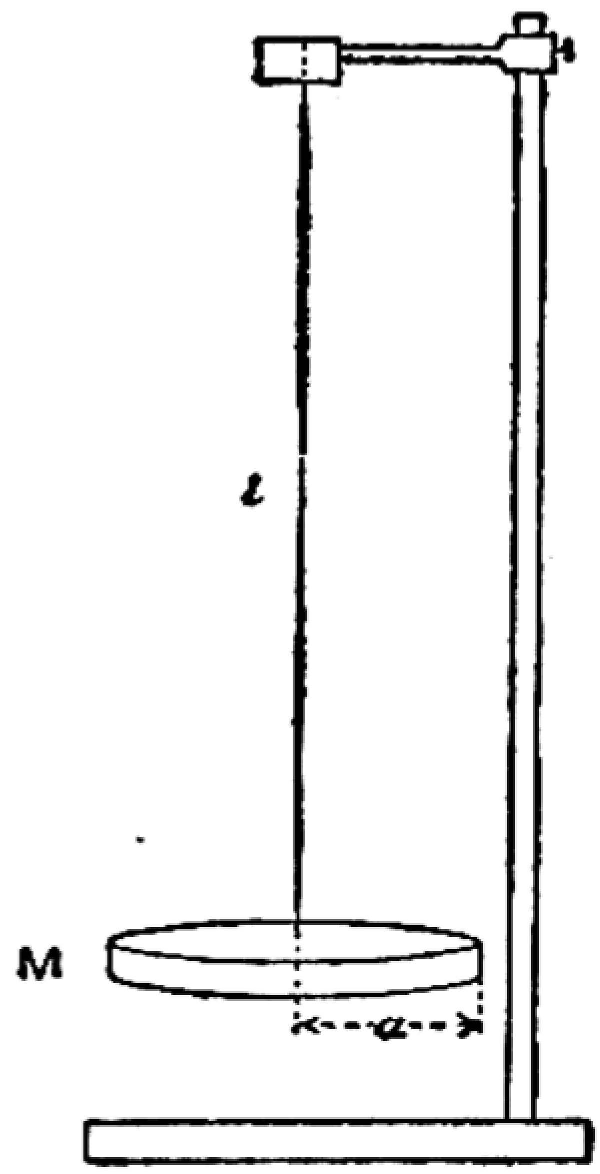

FIG. 79.

measured by the difference in the movements of the two pointers fixed to the rod. The length of the rod is the distance between the points at which the pointers are fixed. A and B are circular degree scales.

(*b*) *Dynamical Method.*—In the case of a wire the rigidity modulus may be found by measuring the time of torsional oscillation of the wire carrying a mass of known moment of inertia at its lower end.

If the suspended mass is a circular disc of mass M and radius a, its moment of inertia $I = \frac{1}{2}Ma^2$.

When the disc is turned about its axis through an angle θ from its equilibrium position, the restoring couple acting on it is that due to the twist θ in the suspending wire. If the wire has length l and radius R, we have $I.\ddot{\theta} = -\frac{\pi n R^4}{2l}.\theta$. The oscillation is simple harmonic and the period $T = 2\pi\sqrt{\frac{2lI}{\pi n R^4}}$.

Thus n is known in terms of T, l, R and I, which can be measured.

It is worthy of notice that this is one of the rare cases in which the motion is simple harmonic for comparatively large amplitudes. If R is small compared with l the maximum shear-angle in the wire will be small compared with θ, and θ may be of the order of a radian or more without passing the elastic limit of the wire.

9. Values of the Moduli

For a homogeneous isotropic substance it can be proved that the following relations hold between the moduli :—

$$n = \frac{E}{2(1+\sigma)} \text{ and } k = \frac{E}{3(1-2\sigma)}.$$

For such materials, since n, k and E are positive, σ must lie between $+\frac{1}{2}$ and -1.

The values of the moduli for a few materials are given below.

Substance	E dynes per sq. cm.	n dynes per sq. cm.	k dynes per sq. cm.	σ
Copper (pure) .	$12{\cdot}3 \times 10^{11}$	$4{\cdot}55 \times 10^{11}$	$13{\cdot}1 \times 10^{11}$	·34
Copper (wire) .	12·7	4·0	14·3	·26
Iron (cast) . .	10–13	3·5–5·3	9·6	·23–·31
Iron (wrought) .	19–20	7·7–8·3	14·6	·27
Steel . . .	20–21	8–9	18	·25–·33
Brass (wire) .	9·7–10·2	3·5	10·7	·34–·40
Quartz Fibre .	5·18	3·0	1·4	
India Rubber .	·05	·00016	—	·48

10. Coefficient of Restitution

The property of matter known as elasticity is responsible for the rebounding of bodies after collision. The first experiments on this subject were made apparently by Newton. He arranged that two spheres of hard material should make 'head-on' collisions and measured their velocities before and after impact. He found that the relative velocity of the spheres after impact bore a constant ratio to their relative velocity before impact. This ratio, the coefficient of restitution, is approximately constant for spheres of the same material and independent of the velocities so long as these are not too great. If the materials are different there is still a definite coefficient of restitution between them independent of the velocities. The same general law is true also if the radius of one sphere is increased indefinitely and it becomes a fixed plane surface.

For direct impact the law may be stated algebraically in the following way. Let u_1, v_1 be the velocities of the first sphere before and after impact respectively in a direction taken as positive; and u_2, v_2 the velocities of the second sphere before and after impact in the same direction, then

$$v_2 - v_1 = -e(u_2 - u_1).$$

The negative sign is introduced because there is a reversal of the relative velocity at impact; *e.g.*, a sphere falling on a fixed surface has the direction of its velocity reversed after the impact.

The coefficients of restitution of different materials are found to be always less than unity and there is always a loss of kinetic energy as a result of the collision.

Newton's law regarding the relative velocities may be verified

and the coefficient of restitution measured by the following experiments.

(*a*) *Impact of Spheres.*—Two metal spheres, not necessarily of the same size and mass, are suspended by long parallel threads so as to touch each other as A, B_1 in Fig. 80. It is convenient

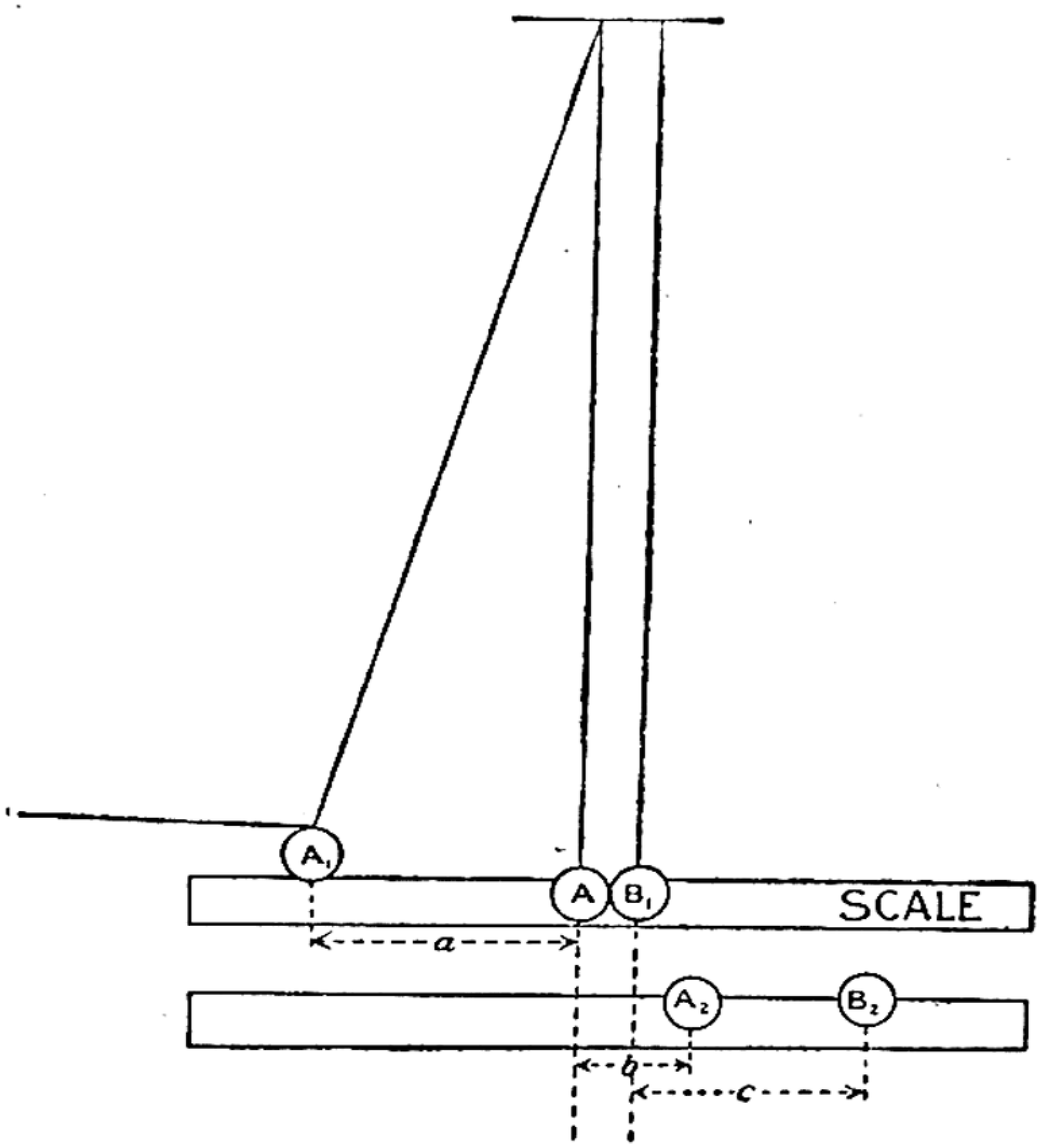

FIG. 80.

if they are furnished with short pointers. A horizontal scale is fixed just behind the spheres. A is drawn aside in the plane of the parallel threads to a position A_1 by a third thread fastened to a fixed support. The distance *a* shown in the diagram is noted and the deflecting thread is burnt. After the collision the spheres swing over to the right and come to rest for an instant. The positions in which they come to rest at the end of this swing are noted and the distances *b* and *c* obtained. The distances *a*, *b*, and *c* are proportional to the velocities of A just before impact, of A just after impact, and of B just after impact. The coefficient

of restitution is $\frac{c-b}{a}$. By starting A at different distances from its rest position Newton's law may be verified.

To show that the velocity at the lowest point of the swing is proportional to the horizontal distance through which a sphere has moved we notice that the path described by the sphere is a circular arc. Let h be the vertical displacement which accompanies a horizontal displacement a. Then $a^2 = h(2\mathrm{L} - h)$, where L is the length of the suspending thread. Now L is large compared with h, so $a^2 = h.2\mathrm{L}$ approximately. But the velocity at the lowest point is given by

$$v^2 = 2g.h = \frac{g}{\mathrm{L}}.a^2.$$

Thus $v \propto a$.

(*b*) *Impact of a Sphere on a Plane Surface.*—A steel ball-bearing is supported by a small electro-magnet above a steel surface-plate. Its height H above the surface-plate D is measured, Fig. 81. On switching off the current the ball drops and the height of rebound h may be measured either by means of a vertical scale fixed close behind the electro-magnet, or by adjusting a retort-stand ring C so that its plane is the same height h above the plate D.

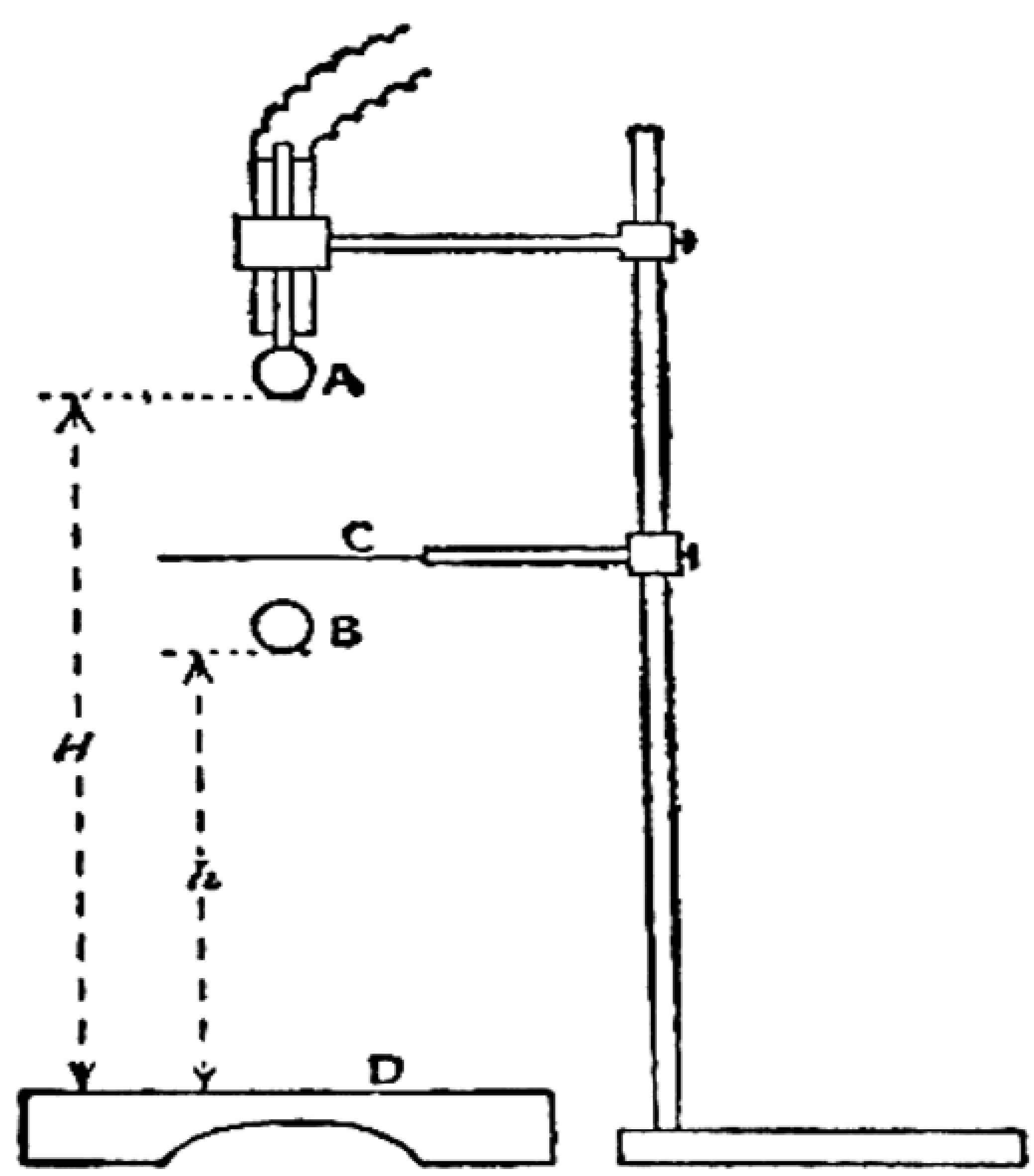

FIG. 81.

The coefficient of restitution, which is the ratio of the relative velocity of ball to plane after impact to the relative velocity before impact, is equal to $\sqrt{\frac{h}{\mathrm{H}}}$.

Attempts have been made to express the coefficient of restitution between spheres in terms of the moduli of elasticity, but the problem is difficult and not of immediate practical interest.

When two spheres collide each suffers a slight deformation with flattening round the point of contact. The elastic forces called into play restore the shape of the spheres and give them

momentum in the opposite direction. Vibrations also are set up in the spheres which are dissipated as heat and sound. Thus the kinetic energy of the spheres after collision must always be less than before impact.

11. Conservation of Momentum

Newton's experiment described in the previous section may be used also to verify the Principle of the Conservation of Momentum. As applied to the spheres in this experiment the Principle states that the total momentum of the spheres just before impact is equal to their total momentum after impact. Here, of course, just before and after impact, the spheres are either at rest or moving along the same line, and the total momentum of the two spheres is found by simple addition. If the masses of A and B (Fig. 80) are M and m respectively, it will be found that

$$M.a = M.b + m.c.$$

The experiment may be modified by drawing both A and B aside, releasing them at the same instant and noting the distances of rebound. (The impact will always take place at the lowest point since the simple pendulum is isochronous.) In this case the spheres will be moving at impact in opposite directions. One direction of motion must be taken as positive and the momentum in this direction as positive, momentum in the opposite direction being reckoned negative.

The Conservation of Momentum Principle states that in an isolated system consisting of a number of particles in motion, colliding and attracting and repelling each other, the total momentum of the system remains constant.

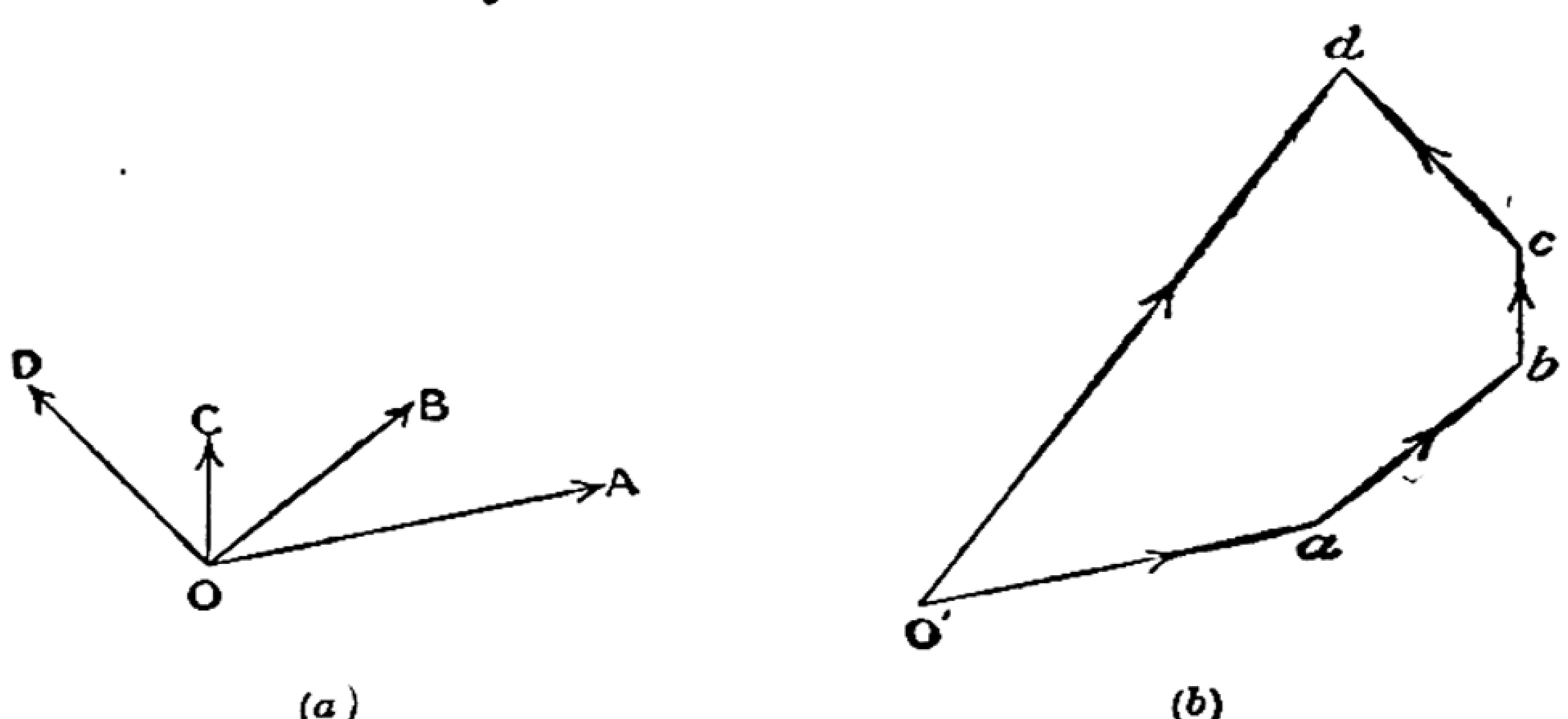

Fig. 82.

Since the velocity of a particle is a vector quantity, its momentum, being mass × velocity, is also a vector, and the total momentum of two particles is given by the parallelogram law for compounding vectors. Further, let the vectors OA, OB, OC, OD represent at some instant the momenta of four particles constituting an isolated system. Draw O′*a*, *ab*, *bc*, *cd* equal and parallel to OA, OB, OC, OD respectively; then O′*d* represents the total momentum of the four particles, and it is this vector which remains invariable. And so for any number of particles.

It is clear, also, that the sum of the momenta of the separate particles resolved in any particular direction is invariable, since it is equal to the total momentum resolved in the same direction. It is in this form that the principle of the conservation of momentum finds most of its applications.

Let us apply these principles to the solution of the problem depicted in Fig. 83. Two smooth spheres, masses 2 lb. and 1 lb.,

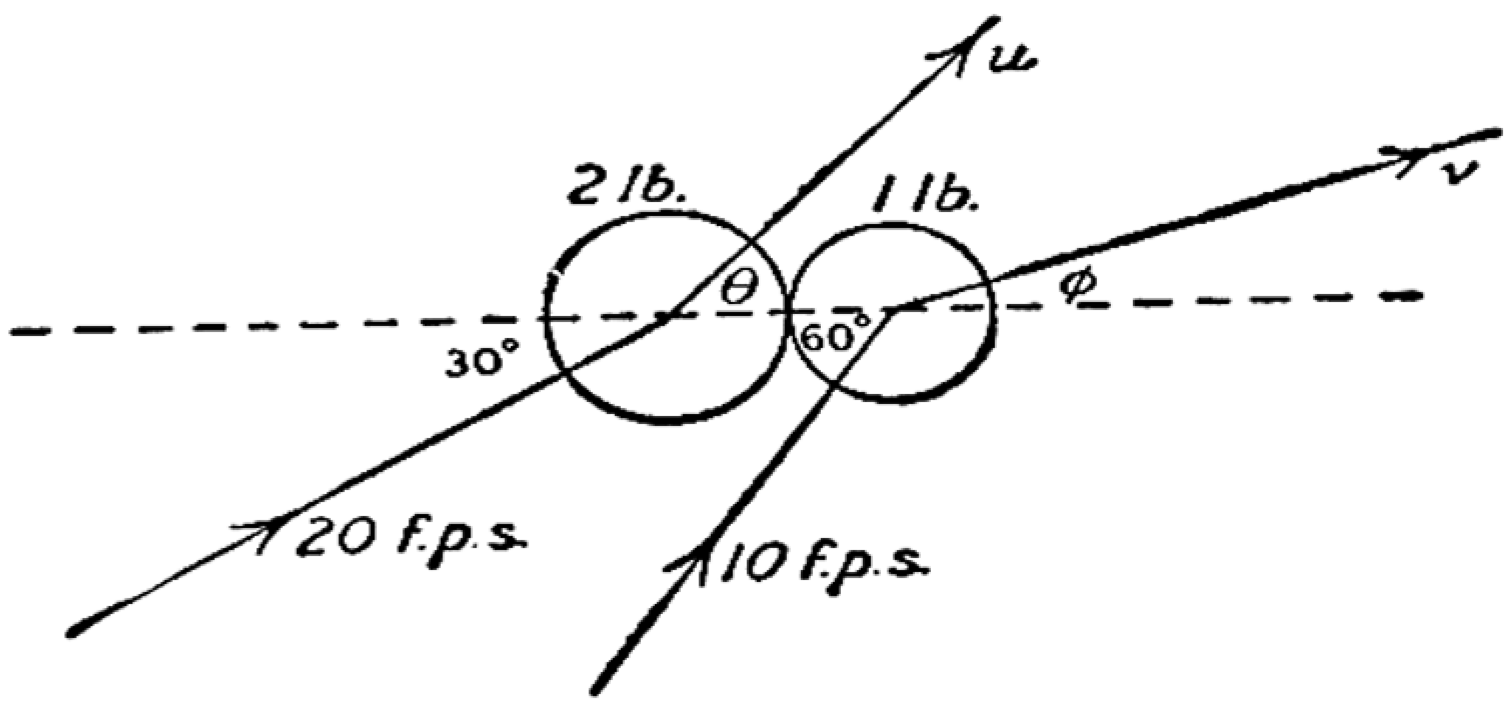

FIG. 83.

with velocities as shown in the diagram, collide so that their directions of motion make angles of 30° and 60° with the line of centres at the instant of collision. To find the velocities, u and v, of the spheres after collision, assuming a coefficient of restitution of ·8.

The action of the impact is along the line of centres and Newton's law applies to the velocities along this line. We have, then,

Final velocity of 1st sphere − Final velocity of 2nd sphere = − e (Initial velocity of 1st sphere − Initial velocity of 2nd sphere)

with the negative sign before the coefficient of restitution, since there is always a reversal of relative velocity at impact.

Thus
$$u \cos\theta - v\cos\phi = -\cdot 8\,(20\cos 30^\circ - 10\cos 60^\circ)$$
$$= -9{\cdot}856 \quad \ldots\ldots\ldots \quad (1)$$

Also, since the spheres are smooth and the only action between them is along the line of centres the velocity of each sphere perpendicular to the line of centres is unchanged.

Whence $$u \sin \theta = 20 \sin 30° = 10 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and $$v \sin \phi = 10 \sin 60° = 8{\cdot}66 \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Further, the momentum along the line of impact is unchanged, so

$$2u \cos \theta + 1v \cos \phi = 2.20 \cos 30° + 1.10 \cos 60°,$$

i.e., $$2u \cos \theta + v \cos \phi = 39{\cdot}64 \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Adding (1) and (4) $$3u \cos \theta = 29{\cdot}78$$

and $$u \cos \theta = 9{\cdot}93.$$

Whence, using (2),

$$u^2 = 10^2 + 9{\cdot}93^2 \text{ and } \tan \theta = \frac{10}{9{\cdot}93},$$

giving $$u = 14{\cdot}09 \text{ f.p.s. and } \theta = 45° \ 12'.$$

Multiplying (1) by 2 and subtracting from (4), we have

$$3v \cos \phi = 59{\cdot}35$$

and $$v \cos \phi = 19{\cdot}78.$$

Whence, using (3)

$$v^2 = 19{\cdot}78^2 + 8{\cdot}66^2 \text{ and } \tan \phi = \frac{8{\cdot}66}{19{\cdot}78},$$

giving $$v = 21{\cdot}59 \text{ f.p.s. and } \phi = 23° \ 38'.$$

Examples

1. State Hooke's law and describe briefly how you would investigate it for the elongation of a copper wire. Draw a graph to show the nature of the results obtained if the experiment is continued until the wire breaks.

A wire extends 2 mm. when a load of 1 kilo is applied. Supposing Hooke's law to hold determine in ergs the work done in stretching the wire when the load is increased gradually from 4 kilos to 5 kilos. (N.U.)

(The work done is the area in the Load-Extension graph enclosed between the curve, the extension axis, and the ordinates representing the initial and final loads. If Hooke's law is obeyed the graph is a straight line and the work is equal to the mean force times the extension between these loads.)

2. Explain the meaning of 'stress' and 'strain.' In what circumstances is their ratio known as Young's modulus? What meaning can be attached to their product?

A uniform wire 300 cm. long, weighing 21·0 gm., elongates 2·4 mm. when stretched by a force of 5·0 kilos wt. The density of the metal is 8·8 gm. per c.c. Determine (*a*) the value of Young's modulus for the metal, (*b*) the energy stored in the wire. State the units in which each result is expressed.

3. Explain what is meant by a 'modulus of elasticity,' illustrating your answer by carefully defining any 2 moduli of elasticity. Describe how you would determine the modulus of elasticity involved in the stretching of a wire, pointing out the precautions necessary to obtain an accurate result.

4. Two vertical parallel wires of the same length and material are fastened together at both ends and are used to support a heavy weight. If the diameter of one wire is double that of the other, what fraction of the weight will each wire support?

5. How would you make an experiment to test Hooke's law in the case where a wire is fixed at one end and twisted by a couple applied at the other end?

A mass of metal of volume 500 c.c. hangs on the end of a wire whose upper end is rigidly fixed. The diameter of the wire is uniform and equal to 0·4 mm. and its Young's modulus is 7×10^{11} dynes per sq. cm. When the metal mass is completely immersed in water the length of the wire changes by 1 mm. Find the length of the wire. (N.U.)

6. If the coefficient of linear expansion of steel is 12×10^{-6} per degree C. and Young's modulus is 2×10^{9} gm. wt. per sq. cm., calculate the increase in tension (gm. wt.) in a steel wire of cross-section 0·005 sq. cm. tightly fixed between two rigid supports when the temperature falls by 10° C.

7. The following observations were made with a wire 5 metres long, having a mean diameter of 0·2 mm.

Load (gms.) . .	10	22	35	48	60	75	81	90
Extension (mm.) .	·16	·32	·56	·77	·96	1·20	1·36	1·74

Determine Young's modulus for the material of the wire and account for any peculiarity in the graph of the readings.

8. What is the meaning of the statement that Young's modulus for copper is $1{\cdot}25 \times 10^{12}$ c.g.s. units? What would be its value in lb. ft. sec. units?

A piece of rubber pressure tubing 30·0 cm. long extends by 0·60 cm. when stretched by a load of 300 gm. If the internal

diameter of the tube is 4·0 mm. and the thickness of wall also 4·0 mm., calculate Young's modulus for the rubber. (N.U.)

9. Define 'coefficient of restitution.' Describe an experiment to determine its value in the case of a steel sphere dropped on to a rigid horizontal steel plate.

A helium atom moving in a straight line with velocity V makes direct impact with a hydrogen atom at rest. Assuming that the two atoms behave like perfectly elastic spheres and that the mass of the helium atom is four times that of the hydrogen atom, calculate (*a*) the velocity of the hydrogen atom after the collision, (*b*) the percentage energy lost by the helium atom in the collision. (N.U.) ('Perfectly elastic' means coefficient of restitution is unity.)

10. State the conditions which determine the velocities after collision of two elastic bodies whose velocities before collision are in the same straight line.

Write in symbols the equations representing these conditions.

A stream of particles, each of mass m gm. moving with velocity u cm. per sec., is impinging normally on a plate which is being moved towards the particles at a constant velocity U cm. per sec. The coefficient of restitution for the collision is e. Find the change in kinetic energy of each particle, and also the force employed to maintain the motion of the plate if n particles impinge on it per second. (N.U.)

(The force required will be equal to the change in momentum of the n particles which meet the plate per second.)

CHAPTER IX

VISCOSITY

1. Introductory

THERE are a number of points of interest in the simple act of stirring a cup of tea (without milk). The stirring imparts a mass-motion to the liquid of a rotatory character. After the stirring has ceased the motion dies away in a short time and the tea resumes once more an appearance of rest. It will be observed that points on the surface at different distances from the centre have different angular velocities, the angular velocity increasing towards the centre ; and that particles in contact with the side of the cup have no motion. The various roughly cylindrical layers are in a state of relative motion and it is the internal friction between them that brings the liquid to rest. This internal friction in a fluid is called viscosity. By its agency the kinetic energy of the mass-motion is converted into heat. That this is the inevitable result of the molecular constitution of the fluid will be explained in a later section.

The position of the tea-leaves during the gradual decay of the motion is at first sight paradoxical. They will usually be found at the bottom of the cup gathered together in a slowly gyrating little heap at the centre. They are evidently of greater density than the tea and should therefore be centrifuged outwards to the circumference (see p. 16, Q. 13). Their position in the centre is due to the fact that the liquid in contact with the base of the cup is at rest and that the angular velocity of the motion increases with the height above the base. The greater centrifugal action in the layers above the base thus superposes on the rotatory motion a current down the side of the cup and in towards the centre of the base, similar in form to the convection current obtained by applying heat at the centre of the base. It is this current which gives the tea-leaves their position.

It should be noticed that there is no loss of kinetic energy due to friction between the liquid and the surface of the cup, since the liquid in contact with the solid is at rest.

The time taken for a liquid to come to rest after stirring is a rough inverse measure of its viscosity.

The property is exhibited in widely varying degree by different liquids. Glycerine and treacle are examples of very viscous liquids, whilst ether and the lighter petrols are more mobile than water. In many cases, too, the same liquid shows a wide variation of viscosity with change of temperature.

It is evident that internal friction or viscosity can exist in a gas.

It appears also to be a property of true solids (*i.e.* solids which are not continuously deformed by the steady application of a small force), as, for example, when they are stressed beyond the yield point. There is evidence of something like viscosity, also, in the decay of the torsional oscillations of a wire when the stress is much below that corresponding to the yield point. The decrement of the amplitude is greater than can be accounted for by air resistance. Solids like pitch and glass should, perhaps, be considered as hard and extremely viscous liquids.

The subject is one of importance in engineering as the rate of flow of both liquids and gases along pipes is determined at low speeds largely by their viscosity.

2. Definition of Viscosity

Newton appears to have formulated the first precise statement of the meaning of viscosity.

Consider two parallel planes AB and CD containing the fluid between them. Consider further that the upper plane AB is moving with a steady velocity V to the right and that the lower plane CD is kept fixed. A tangential force T per unit area will be required to keep AB in motion, and an equal and opposite force T per unit area will be required to keep CD at rest. The fluid actually in contact with AB will move with it, that in contact with CD will remain at rest. Particles in between the planes will move parallel to the planes with constant velocities proportional to their distances from CD. The velocity gradient in the fluid perpendicular to CD will be uniform and equal to $\frac{V}{s}$, s being the distance between the planes.

If we consider a thin layer EF of the fluid parallel to the planes (Fig. 84), it is clear that the fluid above EF is dragging EF to the right with a tangential force T per unit area, and that the fluid below EF is exerting on EF an equal and opposite force T per unit area to the left. This tangential stress T, thus set up at any point, Newton assumed to be proportional to the velocity

gradient at that point. The ratio of the tangential stress to the velocity gradient is called the coefficient of viscosity.

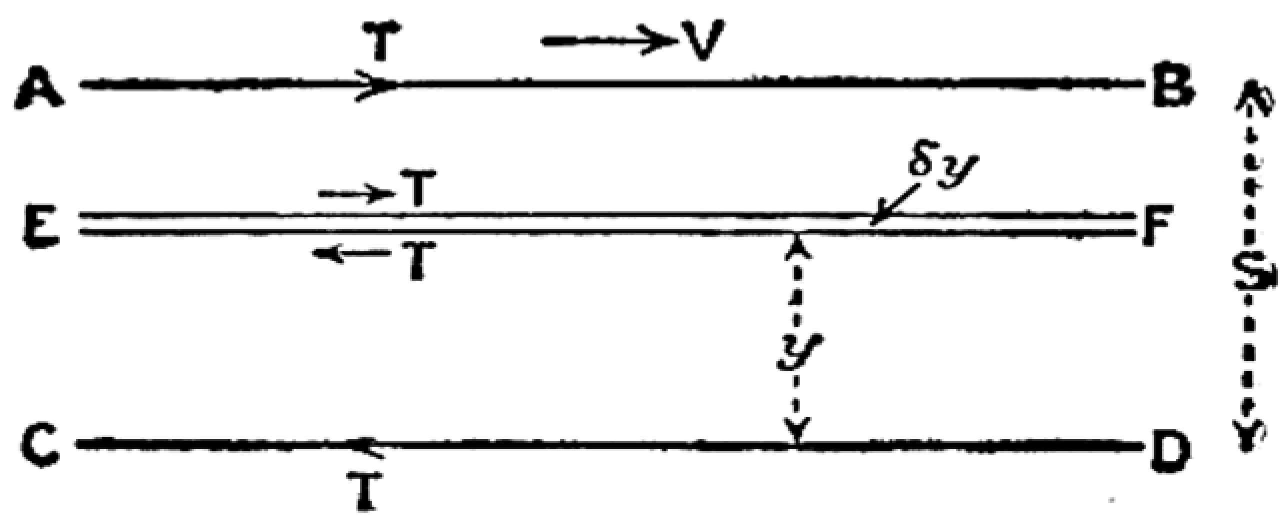

FIG. 84.

If the velocity of the fluid at height y above CD is v, the velocity gradient is $\frac{dv}{dy}$ and η, the coefficient of viscosity, is given by

$$\eta = \frac{\mathrm{T}}{\frac{dv}{dy}}.$$

In the particular case considered above $\frac{dv}{dy} = \frac{\mathrm{V}}{s}$.

The definition of the coefficient of viscosity, or more briefly, of the viscosity, may be stated thus :—

If two parallel planes are at unit distance apart and one is moving parallel to itself with unit velocity relative to the other, then the tangential force per unit area exerted on either plane is equal to the viscosity, the space between the planes being filled with the fluid.

Newton's assumption that the tangential stress is proportional to the velocity gradient is, of course, in agreement with the results of experiment.

The conditions of the definition above are to some extent realised in a form of apparatus devised by Dr. Searle of Cambridge for measuring the viscosity of the more viscous liquids. A cylindrical vessel is pivoted to rotate about its axis, which is vertical, inside a larger fixed coaxial cylinder, the space between them being filled with the liquid. A known couple is applied to the inner cylinder and its speed determined by timing over a few revolutions.

Most of the reliable determinations of viscosity have, however, been made by measuring the rate of flow of the fluid through tubes.

3. Dimensions of Viscosity

The dimensions of velocity gradient are

$$LT^{-1} . L^{-1}, \text{ i.e., } T^{-1}.$$

The dimensions of tangential stress are

$$MLT^{-2} . L^{-2}, \text{ i.e., } ML^{-1} T^{-2}.$$

The dimensions of viscosity are thus

$$ML^{-1} T^{-1}.$$

As an example of the use of dimensions let it be required to convert a viscosity of ·018 c.g.s. units into lb. ft. sec. units, given 1 ft. = 30·5 cm. and 1 lb. = 454 gm.

The dimensional form for viscosity is $ML^{-1} T^{-1}$. To obtain the conversion factor we substitute the ratio of the old to the new unit for the separate letters.

Thus a viscosity of ·018 c.g.s. units

$$= \cdot 018 \times \frac{1}{454} \times \left(\frac{1}{30 \cdot 5}\right)^{-1} \times \left(\frac{1}{1}\right)^{-1}$$

$$= \cdot 018 \times \frac{30 \cdot 5}{454} = \cdot 00121 \text{ lb. ft. sec. units.}$$

$$= \cdot 00121 \text{ pdls. per sq. ft. per unit velocity gradient.}$$

A velocity gradient of 1 cm. per sec. per cm. is the same, of course, as one of 1 ft. per sec. per ft. This is clear from geometrical common sense, but also follows from the fact that the dimensions of velocity gradient are $M^0L^0T^{-1}$.

4. Flow of Liquid through Capillary Tube

If a horizontal capillary tube is fixed near the bottom of a vessel of liquid as in Fig. 85 (*a*) the greater pressure at the end in the vessel will cause a slow steady flow of liquid along the tube. So long as the mean velocity in the tube is below a certain critical value depending on the radius of the tube, the viscosity and density of the liquid, the flow is of the kind known as stream-line. Each particle of the liquid moves with constant velocity parallel to the axis of the tube, particles travelling along the axis having the highest speed and those in contact with the wall of the tube being at rest. Between the axis and the wall of the tube there is a velocity gradient, which, in this case, is not uniform.

Let the radius of the tube be a and its length l. The length is taken to be large compared with the radius, so that end-effects are negligible. Let p_1 and p_2 ($p_1 > p_2$) be the pressures at the

ends of the tube. Consider the motion of the liquid contained between cylinders of radii r and $r + \delta r$ coaxial with the tube. If

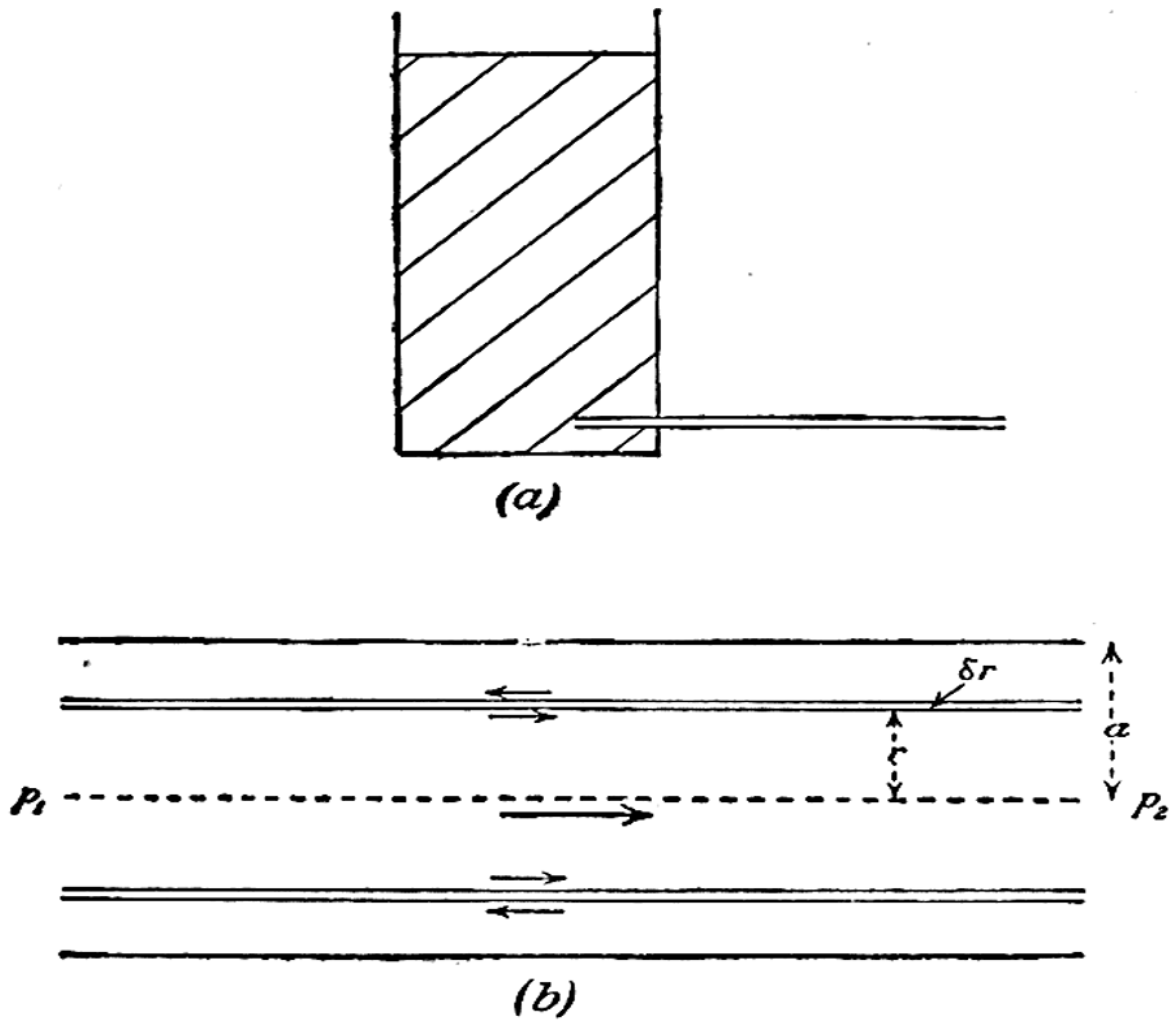

FIG. 85.

η is the viscosity and $\frac{dv}{dr}$ the velocity gradient at distance r from the axis, the inner surface of this cylindrical shell experiences a force per unit area of $-\eta . \frac{dv}{dr}$ in the direction of motion ($\frac{dv}{dr}$ is negative since the velocity decreases as r increases). This gives a total force of $-2\pi\, r.l.\eta\, \frac{dv}{dr}$. On the outer surface of the shell the force opposing the motion will be

$$-(2\pi l\eta r\frac{dv}{dr} + 2\pi l\eta\frac{d}{dr}\left(r\frac{dv}{dr}\right). \delta r).$$

The total force on the shell in the direction of motion is then

$$(p_1 - p_2)2\pi r . \delta r + 2\pi l\eta\frac{d}{dr}\left(r\frac{dv}{dr}\right). \delta r.$$

This is zero since the velocity is uniform, and therefore

$$(p_1 - p_2).r + l\eta\frac{d}{dr}\left(r\frac{dv}{dr}\right) = 0.$$

Integrating, $$(p_1 - p_2)\frac{r^2}{2} + C + l\eta r\frac{dv}{dr} = 0.$$

The constant C is zero, for the equation holds for points on the axis, where $r = 0$, and $\frac{dv}{dr}$ is nowhere infinite.

We have then

$$(p_1 - p_2)\frac{r}{2} + l\eta\frac{dv}{dr} = 0, \quad \ldots\ldots \quad (1).$$

and integrating again,

$$(p_1 - p_2)\frac{r^2}{4} + C' + l\eta\, v = 0.$$

When $r = a$, $v = 0$, so $C' = -(p_1 - p_2).\frac{a^2}{4}$.

Thus $v = \frac{p_1 - p_2}{4l\eta}(a^2 - r^2)$,

giving the velocity at any distance from the axis.

The volume of liquid passing through our cylindrical shell in unit time is $2\pi r.\delta r.v$. The volume Q passing in unit time along the tube is, therefore, given by

$$Q = \int_o^a 2\pi r.\frac{p_1 - p_2}{4l\eta}(a^2 - r^2)dr$$
$$= \frac{(p_1 - p_2)\pi}{2l\eta}\left[\frac{a^2r^2}{2} - \frac{r^4}{4}\right]_o^a$$
$$= \frac{(p_1 - p_2)\pi a^4}{8l\eta}.$$

This equation, Poiseuille's formula, applies only when the flow is stream-line and needs correction if the liquid emerges with appreciable kinetic energy. Poiseuille made many careful experiments on the flow of liquids through tubes, and the close agreement of his results with the above theoretical formula shows that there is no slip of the liquid over the solid boundary and that the liquid in contact with the solid is at rest.

To obtain η in c.g.s. units Q must be in c.c. per sec., a and l in cm., and p_1 and p_2 in dynes per sq. cm.

Equation (1) may be obtained more quickly by considering the forces on a cylinder of the liquid of radius r and length l instead of

those on a cylindrical shell. The whole of the liquid in the cylinder is not moving with the same velocity, but no part of it is accelerated,

so $$(p_1 - p_2)\,\pi r^2 + 2\pi r l \eta \frac{dv}{dr} = 0$$

giving $$(p_1 - p_2)\frac{r}{2} + l\eta\frac{dv}{dr} = 0.$$

5. Stream-line and Turbulent Flow

A series of experiments by Osborne Reynolds has made clear the conditions under which the flow of a liquid along a tube is of stream-line character. The mean velocity, $Q/\pi a^2$, must be below a certain critical velocity, which, with c.g.s. units, is about $1{,}000\eta/\rho a$, a being the radius of the tube and ρ the density of the liquid.

The critical velocity can be found for water in a given tube, and the nature of stream-line and turbulent motions demonstrated with the apparatus depicted in Fig. 86.

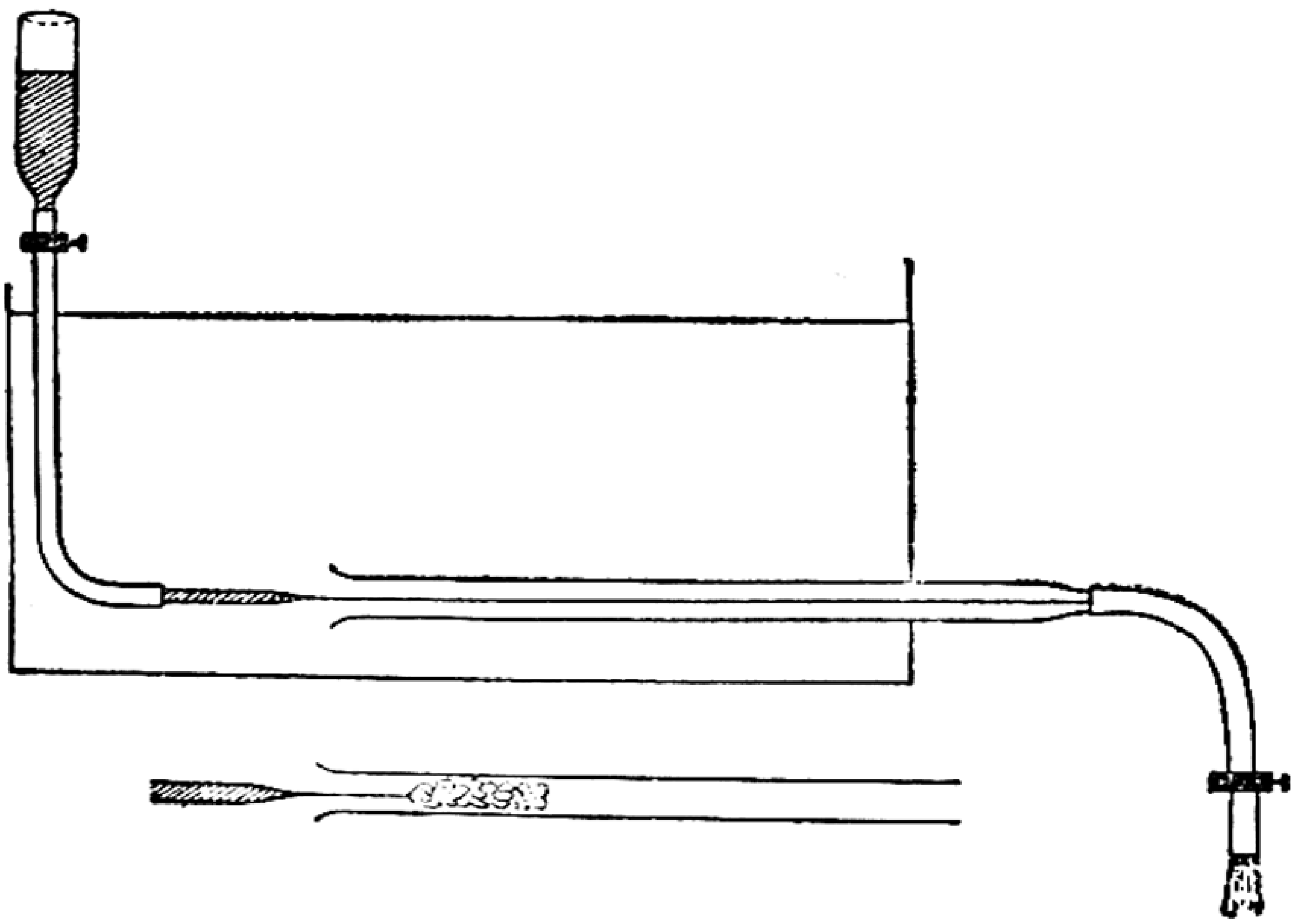

FIG. 86.

Water is contained in the tank and the rate of efflux through the horizontal glass tube can be controlled by means of a clip on a piece of rubber tubing at its end. A small vessel containing ink is connected by thin rubber tubing to a drawn-out glass jet and delivers a fine thread of colour to the stream entering the tube. At low velocities this thread moves along the tube

unbroken and parallel to the axis. As the velocity of flow is increased a critical velocity is reached at which the thread of colour breaks off short and moves from side to side of the tube, eddies are formed and the tube is filled with colour. For water at 20° C. $\eta = \cdot 01$ c.g.s. units and the critical velocity for a tube 1 cm. radius is about 10 cm. per sec. For a tube 1 mm. radius it is 1 m. per sec.

6. Measurement of Viscosity of Water

In all viscosity determinations it is important to keep the temperature constant during the experiment and to record it accurately. Even in the case of water at 15°C. the temperature change of viscosity is about 3 per cent. per °C. With liquids like lubricating oils the change is much greater, particularly if they are not far from the setting temperature.

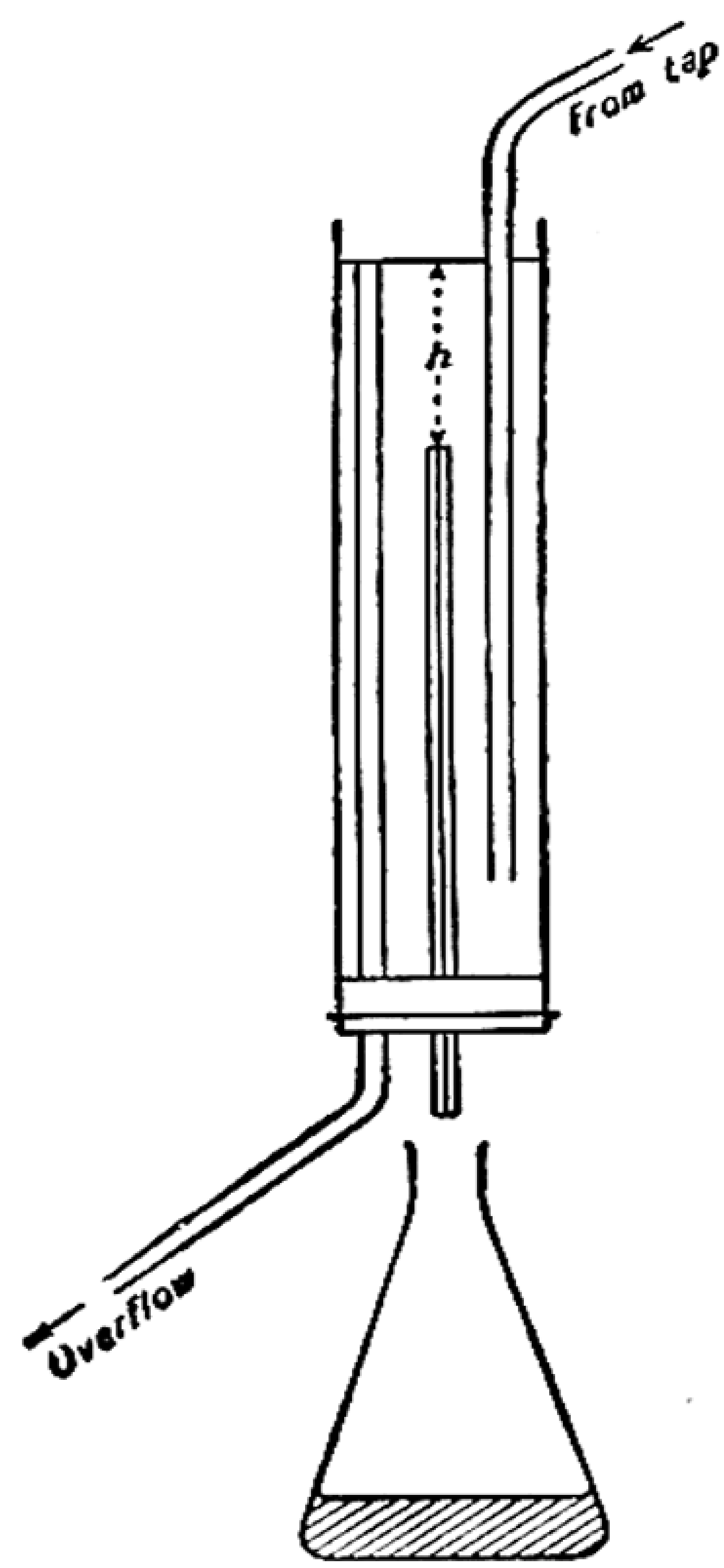

Fig. 87.

The apparatus shown in Fig. 87 may be used for measuring the viscosity of water at room temperature. The slowly running

tap and the overflow pipe give a constant head of water. A thermometer should be suspended in the wide tube alongside the capillary, which could be about 40 cm. long and 1 mm. bore. The water passing down the tube may be caught in a measuring flask and the time to fill up to the mark observed. The length l of the tube and the height h of the water surface above the upper end of the tube must be measured. The radius a of the tube should be determined before the tube is wet, by drawing in a thread of clean mercury, measuring its length and weighing.

Although the conditions are somewhat different from those obtaining in the proof of Poiseuille's formula the modification required is simple. Neglecting surface tension effects the pressure at the bottom of the tube is atmospheric; at the top it is atmospheric $+ h.\rho.g$, where ρ is the density of the water. In addition, since there is no acceleration, the viscous forces are supporting the weight of the column of liquid in the capillary. This is equivalent to a pressure difference between the ends,

$$p_1 - p_2 = (h + l)\rho g.$$

We have then for the volume delivered in t secs.,

$$Q = \frac{\pi a^4}{8l\eta}(h + l)\rho g . t,$$

from which η can be found.

7. Stokes's Law

The viscosity of a liquid like glycerine may be measured at atmospheric temperature by using a formula due to Sir George Stokes. The law states that the force resisting the motion of a sphere of radius a through an unbounded medium of viscosity η is $6\pi\eta av$, v being the velocity of the sphere. The velocity must be small compared with the value of $\frac{\eta}{\rho a}$, ρ being the density of the medium. The expression for the resistance holds only so long as the flow past the sphere is of stream-line character.

If a sphere of radius a and density σ ($\sigma > \rho$) be released in the medium and allowed to fall under gravity, its equation of motion will be

$$\frac{4}{3}\pi a^3\sigma . \ddot{x} = \frac{4}{3}\pi a^3(\sigma - \rho)g - 6\pi\eta a . \dot{x}.$$

It will begin to move with the acceleration $g(1 - \frac{\rho}{\sigma})$ since initially $\dot{x} = 0$, but the acceleration will diminish as the resistance

increases with increasing speed. When the resistance becomes equal to the weight of the sphere less the upthrust of the medium the acceleration will be nil and the sphere will then proceed with a constant velocity, known as the terminal velocity.

We then have $6\pi\eta a v = \frac{4}{3}\pi a^3(\sigma - \rho)g$, v being the terminal velocity; or

$$\eta = \frac{2}{9}.\frac{a^2 g(\sigma - \rho)}{v}.$$

Raindrops fall with a constant terminal velocity, but this will not be given by Stokes's law as their speed is much greater than the critical speed. Water drops falling in air must have a diameter less than ·01 cm. to come under Stokes's law.

For the experiment on glycerine steel ball-bearings may be used, of not more than $\frac{1}{8}''$ in diameter, and the glycerine should be contained in as wide a glass jar as possible. The sphere will fall a little more slowly in the jar than in an infinite medium owing to the influence of the sides, and a correction to the velocity observed is required which becomes important unless the diameter of the ball is very small compared with that of the vessel. Horizontal strips of paper may be pasted front and back of the jar at equal heights to mark a measured track over about the middle third of the jar. The passage of the ball between the marks may be timed by stop-watch. A number of similar balls should be available. Their mean density and mean volume may be found by use of the specific gravity bottle, and from the volume of a definite number the mean radius may be calculated. A number of the spheres should be timed through the glycerine and the mean velocity used with the mean radius. The density of the glycerine has also to be found.

8. Flow of Gas through Capillary Tube

Poiseuille's formula, as given in section 4, cannot, of course, be applied to the flow of a gas, unless the pressure difference, $p_1 - p_2$, is small. A liquid is assumed to be incompressible and there is no change of density as the liquid passes along the tube to lower pressures. In the case of a gas as the pressure decreases its volume will increase, and the speed at all distances from the axis must also increase, since the same mass must pass all cross-sections of the tube in any given time.

We can, however, apply Poiseuille's formula to a short length δx, of the tube over which the pressure change is $-\delta p$.

We have $q = -\dfrac{\pi a^4}{8\eta}\cdot\dfrac{\delta p}{\delta x}$,

where q is the volume entering this short section per sec., the pressure in it being p.

Let P_1, V_1 denote the pressure at the entrance of the tube, and the volume entering per sec.; whilst P_2, V_2 denote similar quantities at the exit end.

Then $$P_1 V_1 = P_2 V_2 = pq,$$

and, therefore, $$P_1 V_1 = -\frac{\pi a^4}{8\eta}\cdot\frac{pdp}{dx}.$$

So $$P_1 V_1 \int_o^l dx = -\frac{\pi a^4}{8\eta}\int_{P_1}^{P_2} pdp,$$

and $$P_1 V_1 l = \frac{\pi a^4}{16\eta}(P_1^2 - P_2^2),$$

or $$\eta = \frac{\pi a^4}{16 l}\cdot\frac{P_1^2 - P_2^2}{P_1 V_1}.$$

If $P_1 - P_2$ is small and $P_1 + P_2$ is nearly equal to 2 P_1, this reduces to Poiseuille's expression.

9. Measurement of Viscosity of a Gas

The apparatus depicted in Fig. 88 is arranged for measuring the viscosity of air at atmospheric temperature, but can easily be adapted for filling with other gases. An inverted graduated burette A is fixed in a stand and connected as shown to a water or oil gauge H, a drying bottle, and the capillary tube CD, about 50 cm. long and about ·5 mm. bore. The tube B is nearly filled with water and is suspended by two threads over two ball-bearing pulleys, the weight of the tube being balanced by the weights PP.

If two extra weights, RR, are added to PP and the burette tap is opened, the tube B will rise, compressing the air in A and forcing it slowly through the capillary. As the air is driven from A, B will continue to rise slowly. In order to compensate for the extra down-thrust on B with greater immersion of the glass of the burette, and so to keep the pressure in A contant, two light chains may be fixed by one end to the pulleys so as to unwind as B rises. The weight of the chains and the radius at which they are fixed must be determined from the volume of glass for which they compensate.

The volume of air driven through the capillary in a given time

can be found from the readings of the water level in A. This volume is measured at the pressure of the atmosphere $+ h$ cm. of

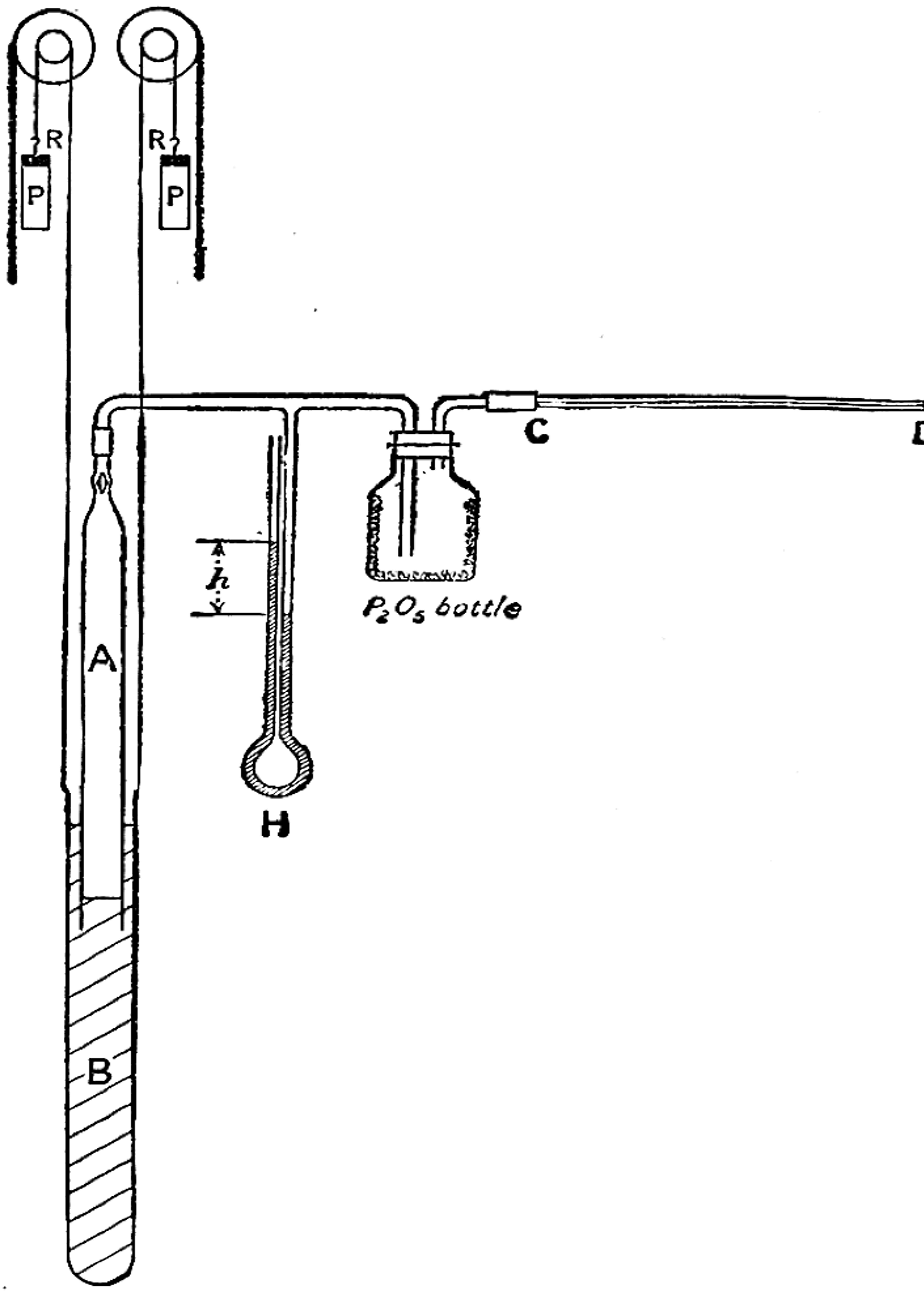

FIG. 88.

water. The height h may be measured with a small cathetometer. The pressure at the open end of the tube is atmospheric. The bore of CD must be found by filling with mercury and

weighing in the ordinary way. The length of CD is also required. The viscosity can then be found from

$$\eta = \frac{\pi a^4}{16l} \cdot \frac{P_1^2 - P_2^2}{P_1 V_1},$$

V_1 being the volume entering the tube per sec., P_1 being the pressure at C in dynes per sq. cm., and P_2 the atmospheric pressure in the same units. If h is less than 10 cm. of water and an error of the order of 1 per cent. is not objected to, the simpler formula of Poiseuille may be used,

$$\eta = \frac{\pi a^4}{8l} \cdot \frac{P_1 - P_2}{V_1}.$$

For measuring the viscosity of hydrogen Lehfeldt has used a method in which the gas was liberated by electrolysis of water. The rate of evolution of the gas (and therefore its pressure) was controlled by the current, and calculated from the strength of the current and the known electro-chemical equivalent of hydrogen.

10. Viscosity and the Kinetic Theory of Gases

Some light is thrown on the nature of viscous forces and the manner in which they arise by the kinetic theory of gases. In this theory the molecules of a gas are considered to be small perfectly elastic spheres in swift and incessant motion. Although the number of molecules per c.c. is assumed to be enormous at atmospheric pressure, the average distance between them at any instant is still large compared with their diameters. The molecules are continually in collision with each other, but it is assumed that for any particular molecule the time spent in collision is small compared with the time between collisions, and that except during collisions the forces between molecules are negligible. As a result of the multitudinous collisions which occur molecules are found moving in all directions with velocities which range from very small to very large. A large proportion of the molecules, however, have velocities which do not differ greatly from the mean velocity, the number with extreme velocities being comparatively small. This condition of random motion is called molecular agitation.

A complete mathematical treatment of a system of this kind is difficult, but fairly accurate results can be obtained by assuming one-sixth of the number of molecules to be moving in each of the directions, positive and negative, of three rectangular axes, all

with the same velocity equal to the average velocity of the molecules.

A gas thus exerts pressure by maintaining a continual bombardment of the walls of the containing vessel, the momentum of a molecule perpendicular to the wall being reversed at impact. If we imagine the containing vessel to be a centimetre cube, the change in momentum produced in one second by one face of the cube will be equal to the force exerted by that face on the gas, and this will be equal and opposite to the pressure of the gas. If n is the number of molecules per c.c., the number moving towards one face of the cube may be taken as $\frac{n}{6}$. If c is their velocity and m the mass of a molecule, the change in momentum at a single impact is $2mc$. The number reaching the face in 1 sec. is the number moving in the right direction contained in a length c, *i.e.*, $\frac{n}{6}.c$. The pressure p is, therefore, $\frac{n}{6}c \times 2mc = \frac{1}{3}\,nmc^2$.

The product $n.m = \rho$, the density of the gas, and

$$p = \tfrac{1}{3}.\rho.c^2.$$

This equation is equivalent to Boyle's law. The 'average' velocity of the molecules may be calculated from it.*

For a more complete account of the kinetic theory and a discussion of the deductions to be made from the equation $p = \frac{1}{3}\,n.m.c^2$, the student should consult text-books on Heat or the Kinetic Theory of Gases.

The average distance travelled by molecules between collisions is called the 'mean free path.' An estimate of the mean free path may be made as follows. A collision occurs when the centres of two molecules approach to a distance σ, equal to the molecular diameter. If we imagine one particular molecule surrounded by a sphere of diameter 2σ, collision will occur with this molecule whenever the centre of another enters this sphere. In one second this sphere, of diameter 2σ, will sweep through a volume $\pi\sigma^2.c$, which will contain $n\pi\sigma^2 c$ molecules. This is the number of collisions in a length of path c. The mean distance between collisions is thus $\frac{c}{n\pi\sigma^2 c}$, and the mean free path $L = \frac{1}{n\pi\sigma^2}$.

* The equation $p = \frac{1}{3}\,\rho c^2$ is exact if c^2 is the mean of the squares of the velocities of the molecules; c is about 9 per cent. bigger than the arithmetic mean of the velocities.

This result is, of course, only approximate. The actual value of L is close to $\frac{\cdot 7}{n\pi\sigma^2}$.

We are now in a position to calculate the viscosity of the system. Consider two plane horizontal surfaces 2 cm. apart with the gas between them, the upper surface moving to the right with a velocity of 1 cm. per sec., the lower one to the left with the same speed. The velocity gradient in the gas will be 1 cm. per sec. per cm., and the molecules in the central plane will have no mass-motion. Owing to the molecular agitation molecules will be continually passing through this central plane from the upper half to the lower half and an equal number will be passing in the opposite direction. The upper half of the gas is thus continually losing momentum to the right and gaining momentum to the left. This is, of course, equivalent to a tangential force acting on it, opposing its motion to the right. The transfer of horizontal momentum per sec. through 1 sq. cm. of the central plane will be equal to the tangential force per sq. cm., and this, since the velocity gradient is unity, is equal to the viscosity. The number of molecules passing downwards per sec. through 1 sq. cm. of the central plane will be $\frac{1}{6}nc$. These have come on the average from a last collision at distance L, the mean free path, above the central plane, and carry therefore, each of them, a horizontal momentum mL to the right, since the velocity of the mass-motion at a distance L above the central plane is L cm. per sec. The number of molecules passing upwards per sec. through this 1 sq. cm. is also $\frac{1}{6}nc$, each carrying horizontal momentum mL to the left.

The result is a loss of right momentum by the upper half of the gas of amount $\frac{1}{3}ncm$L per sec. per sq. cm.

Thus
$$\eta = \tfrac{1}{3}nmc\mathrm{L} = \tfrac{1}{3}\rho c\mathrm{L}.$$

Since
$$\mathrm{L} = \frac{1}{n\pi\sigma^2}$$

we have
$$\eta = \frac{mc}{3\pi\sigma^2},$$

in which the numerical factor is a little in error.

Viscosity is here revealed as a consequence of the molecular agitation. The kinetic energy of the mass-motion is gradually transformed by collisions of the molecules into increased kinetic energy of molecular agitation, thus the heat-content and temperature of the gas rise as the mass-motion disappears.

In liquids there is a similar molecular agitation, but the molecules are so close together that they are always under the influence of their immediate neighbours and free paths in the sense above cannot exist. The effect of continual collisions, however, must be the same as in gases, the transformation of energy of mass-motion into that of molecular agitation.

The following figures may give a more definite picture of the condition of a gas.

The number of molecules of nitrogen in 1 c.c. at 0° C. and 1 atmosphere pressure is $2{\cdot}7 \times 10^{19}$. The mean distance between them is

$$(2{\cdot}7 \times 10^{19})^{-\frac{1}{3}} = 3{\cdot}3 \times 10^{-7} \text{ cm.}$$

The mean free path is $9{\cdot}4 \times 10^{-6}$ cm., considerably larger, of course, than the mean distance apart. The molecular diameter is about 3×10^{-8} cm.

The mean molecular velocity is $4{\cdot}5 \times 10^{4}$ cm. per sec. A single molecule makes about 5×10^{9} collisions per sec.

At a pressure of one millionth of an atmosphere, which would be a very good vacuum, the number of molecules per c.c. at 0° C. is still $2{\cdot}7 \times 10^{13}$ and their mean distance apart is of the order of a ten-thousandth of a millimetre, whilst the mean free path is something like 10 cm.

11. The Effect of Pressure on the Viscosity of a Gas.

From the expression for the viscosity in terms of molecular magnitudes, $\eta = \dfrac{mc}{3\pi\sigma^2}$, Maxwell deduced that the viscosity of a gas should be independent of its pressure, temperature being constant, since the three quantities involved m, c, and σ are unaffected by change of pressure. This rather surprising result was immediately verified by experiment for pressures from several atmospheres to a few millimetres of mercury. At very low pressures the viscosity decreases as the pressure decreases. It ceases to be constant when the mean free path of the molecules becomes of the same order of magnitude as the dimensions of the space occupied by the gas under experiment.

Maxwell's determinations of the viscosity of gases at different pressures consisted of measuring the damping of the rotational oscillations of a circular disc suspended by a wire through its centre half-way between fixed parallel plates at a small distance apart. The motion of the disc is opposed by a couple due to viscosity which is proportional at any instant to the angular velocity of the disc. The calculation of the viscosity from the observed decrement of amplitude of the oscillations is difficult owing to the edge-effect, but to prove that the viscosity was

independent of the pressure it was sufficient to show that the damping remained the same when the pressure was changed.

At high pressures the viscosity of a gas increases with pressure.

12. Viscosity and Temperature

From the same expression, $\eta = \frac{mc}{3\pi\sigma^2}$, Maxwell concluded that, for a gas, the viscosity should be proportional to the square root of the absolute temperature. This prediction was less successful. Experiment showed that the viscosity increased at a greater rate with rise of temperature than Maxwell predicted. Of the quantities in the formula, m is constant and c varies as $T^{\frac{1}{2}}$, T being the absolute temperature. Maxwell assumed σ to be constant also. This, of course, would be very nearly true for the perfectly elastic hard spheres of the kinetic theory, but the molecules of a gas are not hard spheres and the collisions between them differ in several respects from those of hard spheres. In collisions between molecules moving with high speeds the centres of the molecules come closer together than at lower speeds. As the temperature of a gas is raised the average speed of the molecules increases and the molecular diameter therefore decreases, since this is merely the average distance between the centres of molecules in collision.

It is found experimentally that η varies approximately as T^n, where n lies between ·7 and ·9 for different gases.

Rise of temperature has the opposite effect on the viscosity of liquids. There is always a considerable decrease of viscosity particularly at temperatures near the solidifying point, but at higher temperatures the change per degree rise becomes smaller. The extent of this temperature effect will be seen for a few liquids from the table below.

Table of Viscosities (c.g.s. units).

Substance	Temperature 0° C.	20° C.	100° C.
Air . . .	·000170	·000184	·000220
Hydrogen . .	·000086	·000090	·000106
Water. . .	·018	·010	·0028
Ether . . .	·0029	·0023	
Glycerine . .	46·0	8·5	
Mercury . .	·017	·016	·012

The effect of pressure on viscosity varies with the liquid. Some liquids have their viscosity increased by rise of pressure, in other cases the viscosity is decreased. With liquids of low viscosity, however, the change for moderate pressures is small.

The viscosities of the other simple gaseous substances are of the same order of magnitude as that of air. Hydrogen has the lowest viscosity.

Examples

1. How is the coefficient of viscosity defined? Describe fully a method by which the viscosity of a liquid may be determined.

2. What are the dimensions of the coefficient of viscosity? How would you compare experimentally the viscosities of hydrogen and air at atmospheric temperature?

3. Calculate the viscosity of water from the following results obtained for the flow through a horizontal capillary tube: steady head of water = 30 cm.; volume delivered in 15 mins. = 40 c.c.; length of tube = 50 cm.; a thread of mercury, S.G. 13·6, which occupied a length 42·6 cm. of the tube weighed 2·906 gm.

4. Estimate the volume of air at atmospheric pressure which will pass per sec. into a good vacuum from the outside atmosphere through a capillary tube 20 cm. long and 0·2 mm. bore, taking $\eta = 0{\cdot}00018$ c.g.s. units and 1 atmosphere $= 10^6$ dynes per sq. cm.

5. Two tubes open at both ends, each 30 cm. long, bores 2 mm. and 0·1 mm. respectively, are joined by a piece of rubber tubing to form a U-tube, both limbs being vertical. Near the top of the wider tube is a thread of mercury 5 cm. long, which is observed to fall at the rate of 6 cm. per min. What value does this give for the viscosity of air? (At. pressure = 76 cm. of mercury, S.G. of mercury = 13·6).

6. Two equal circular discs A and B, of radius a, are placed in air parallel to each other at a small distance d apart, with their axes in line. The disc A is caused to rotate with uniform angular velocity ω; find the couple required to keep B at rest, neglecting any edge-effect, the viscosity of air being η.

7. Find the radius of water drops which fall in air at the steady speed of 20 cm. per min. (η for air $= 0{\cdot}00017$ c.g.s. units, density of air $= 0{\cdot}0013$ gm. per c.c.).

8. Find the terminal speed of an air-bubble of radius 1 mm. in glycerine of density 1·26 gm. per c.c. and viscosity 10·0 c.g.s. units.

9. A small drop of oil, S.G. 0·9, is observed to fall in air at the steady rate of 1 mm. per sec. If it acquires a charge equal to that of 20 electrons, what strength of electric field will just prevent the drop from falling? ($\eta = 0{\cdot}00018$ c.g.s. units, density of air $= 0{\cdot}0013$ gm. per c.c., electron charge $= 4{\cdot}77 \times 10^{-10}$ E.S.U.).

10. Describe an experiment to demonstrate the difference between stream-line and turbulent flow. Explain how you would determine experimentally the critical speed at which the change from stream-line to turbulent flow takes place for water passing through a given tube, stating clearly what measurements would be necessary.

11. Gas is contained in a vessel at pressure P_1 (above atmospheric) and allowed to leak into the atmosphere through a long fine capillary tube until the pressure in the vessel is P_2. Show that the times taken for different gases are in the ratio of their viscosities.

ANSWERS

CHAPTER I

1. 1·46, 1·03, 0·843, 0·730, 0·653 radians per sec. 0·0938, 0·1876, 0·281, 0·375, 0·469.
2. 154·6.
4. 2·44″, 0·051 ton wt.
5. 50·6 f.p.s., $\mu > \cdot 8$.
6. 10° 23′.
9. 193·6 ft.
10. 11·3.
11. $T_1 = 3\ mg\ ;\ T_2 = mg.\dfrac{3l - a}{l - a}$.
14. $\mathrm{Tan}^{-1}\dfrac{\omega^2 . r}{g}$.

CHAPTER II

1. 10·88 cm. per sec.
2. 25 f.p.s.
4. 3·174 sec.
5. If acceleration is a when displacement is b,

$$T = 2\pi\sqrt{\frac{b}{a}}:\quad 0{\cdot}898 \text{ sec.}$$

6. 0·453 sec., 1·156 f.p.s.
8. 2·7 sec.
9. 1·053 sec., 0·953 sec.
11. 5° 36′.
12. Rel. vel. is $v \cos \dfrac{v}{a}.t$.

CHAPTER III

1. $M\dfrac{b^2}{3}$.
2. $\dfrac{7}{48}Ma^2$, a being length of rod. 243 ergs.

3. $1:2$.
4. $21^\circ 7'$.
6. $\bar{1}\cdot 64$ secs.
10. $l/\sqrt{3}$.

CHAPTER IV

7. $1{\cdot}33 : 1$.
8. $6{\cdot}9 \times 10^{-8}$.
9. 62·7 mins.
10. $1 : 9{\cdot}29$.
12. $2{\cdot}01 \times 10^{33}$ gm.
14. 3 hrs. 18 mins.
15. $10{\cdot}2 \times 10^{5}$ cm. per sec.
16. $\frac{3}{5}\mathrm{G.M}^2$.

CHAPTER V

1. 20·8 lb. wt., 176·7 lb. wt.
2. 2 ft. from surface, or 1·522 ft. taking atmospheric pressure into account. 18,984 lb. wt.
3. 613 ft.
4. $9{\cdot}81 \times 10^{8}$ dynes, $4{\cdot}9 \times 10^{8}$ dynes.
6. 255·5 gm. tin, 44·5 gm. copper.
7. 0·9.
8. 15·2 cm. of mercury.
9. Atmospheric pressure changed from 76 cm. to 74·9 cm.
10. 4 minutes.
13. 316·8 cm. per sec. per sec.

CHAPTER VI

2. $\mathrm{P} + h\rho g + \frac{2\mathrm{T}}{r}$ dynes per sq. cm.

   $(\mathrm{P} + h\rho g)r^3 + 2\mathrm{T}r^2$ is constant.
3. $\frac{4^3}{6^3} \times \frac{1{,}000{,}030}{1{,}000{,}020}$.
4. 0·68 gm. per c.c.
5. 5·73°.
8. 1·45 cm.
10. 39·2 dynes per cm.
11. 38,300 ergs.

CHAPTER VII

1. 0·204.
2. 3·96 : 1.
4. 13 ft.
5. 0·414 μW.
6. 0·577.
7. $\cot\theta - 2\cot\phi = \mu$.
8. $\dfrac{b}{2\mu}$.
9. When distance between handles $= \dfrac{a}{\mu}$, where a is breadth of drawer from front to back.
10. $\dfrac{M(\mu \cos a + \sin a)}{\cos a - \mu \sin a}$ lb. wt.
12. (1) 10° 8′. (2) 12° 48′.
13. 80 ft. per sec., 51° 20′.

CHAPTER VIII

1. $8{\cdot}83 \times 10^5$ ergs.
2. $7{\cdot}7 \times 10^{11}$ dynes/sq. cm., $5{\cdot}88 \times 10^5$ ergs.
4. $\dfrac{4}{5}$ and $\dfrac{1}{5}$.
5. 179·4 cm.
6. 1,200 gm. wt.
7. $9{\cdot}74 \times 10^{11}$ dynes/sq. cm.
8. $8{\cdot}41 \times 10^{10}$ pdls./sq. ft., $1{\cdot}46 \times 10^7$ dynes/sq. cm.
9. $\dfrac{8V}{5}$, 64 per cent.
10. $\dfrac{1}{2}m(e + 1)\left\{e(u + U)^2 + U^2 - u^2\right\}$ ergs, $nm(e + 1)(u + U)$ dynes.

CHAPTER IX

3. 0·0133 c.g.s. units.
4. 0·545 c.c.
5. 0·00018 c.g.s. units.
6. $\dfrac{\pi\eta\omega a^4}{2d}$.
7. $5{\cdot}1 \times 10^{-4}$ cm.
8. 0·272 cm. per sec.
9. 10·78 E.S.U. or 3,234 volts per cm.

INDEX

Title [redacted]

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Borrower's No.	Issue Date	Borrower's No.	Issue Date

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